Definition – $P_{m,k}$ is called a Poincaré series of weight k. If $m=0$ then
$$E_k = P_{0,k}$$
is called an Eisenstein series of weight k.

Our next task will be to show that $P_{m,k}$ and E_k are in fact enough to describe all holomorphic modular forms subject to some growth conditions.

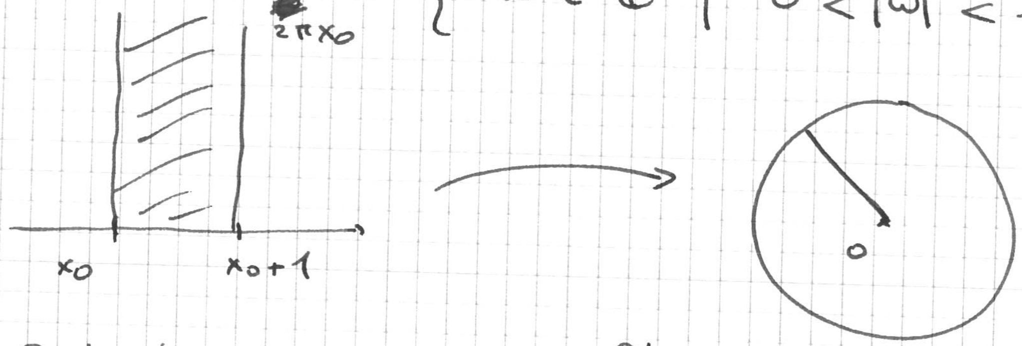
4 – Fourier expansions

We will next explore an extremely useful way of studying modular forms, which exploits the fact that $SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ has a distinguished "direction to infinity"; the behavior of functions in this direction reveals a lot of information.

Proposition (Fourier expansion at infinity)

Let f be a meromorphic modular form of weight k. There exists a meromorphic function $\tilde{f} : D^* \longrightarrow \mathbb{C}$ such that
$$f(z) = \tilde{f}(e^{2i\pi z})$$
for $z \in \mathbb{H}$, where $D^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$.

__Proof.__ Observe that ~~[illegible]~~ the strip
$$\tilde{D}_{x_0} = \{ z \in \mathbb{H} \mid x_0 < \mathrm{Re}(z) < x_0 + 1 \}$$
is mapped bijectively and conformally by $z \mapsto e^{2i\pi z}$ to the set
$$D_{2\pi x_0} = \{ \omega \in \mathbb{C} \mid 0 < |\omega| < 1 ,\ \arg(\omega) \neq 2\pi x_0 \}$$



Restricting f to (say) $\tilde{D}_0$ gives a function $\tilde{f}_0 : D_0 \longrightarrow \mathbb{C}$, meromorphic, such that $f(z) = \tilde{f}_0(e^{2i\pi z})$ for $z \in \tilde{D}_0$. Similarly, restricting to $\tilde{D}_{-1/2}$ gives $\tilde{f}_{-1/2} : D_{-1/2} \longrightarrow \mathbb{C}$ s.t. $f(z) = \tilde{f}_{-1/2}(e^{2i\pi z})$. ~~[illegible]~~ Since f is 1-periodic we get $\tilde{f}_0 = \tilde{f}_{-1/2}$ on $D_0 \cap D_{-1/2}$, hence these give a meromorphic extension $\tilde{f} : D^* = D_0 \cup D_{-1/2} \longrightarrow \mathbb{C}$.

$\square$

__Definition.__ A meromorphic modular form f of weight k is called $\left\{ \begin{matrix} \text{meromorphic} \\ \text{holomorphic} \end{matrix} \right\}$ at ∞ if $\tilde{f}$ has this property at $0 \in D^*$, i.e. if $\tilde{f}$ extends to a $\left\{ \begin{matrix} \text{meromorphic} \\ \text{holomorphic} \end{matrix} \right.$ $\tilde{f} : D \longrightarrow \mathbb{C}$, where $D = \{ \omega \in \mathbb{C} \mid |\omega| < 1 \}$.

We denote by $\widehat{M}_k$ the space of meromorphic modular forms of weight k, meromorphic including at infinity, by M_k the subspace of those holomorphic (also at ∞), by $M_k^0 \subset M_k$ the subspace of those hol. f such that $\tilde{f}(\infty) = 0$; these are called "cusp forms".

Let $f \in \tilde{M}_k$. By ~~definition~~ Laurent expansion of $\tilde{f}$ at 0, we find complex numbers $(a_n)_{n \in \mathbb{Z}}$, with $a_n = 0$ if n is negative enough ($n \leq n_0$ for some n_0) such that

$$\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n \quad , \quad q \in D^*$$

Equivalently:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2i\pi n z} = \sum_{n \in \mathbb{Z}} a_n \, e(nz).$$

We have $\begin{cases} f \in M_k \iff a_n = 0 \text{ for } n \leq -1 \\ f \in M_k^0 \iff a_n = 0 \text{ for } n \leq 0 \end{cases}$

<u>Definition</u>. The $(a_n)_{n \in \mathbb{Z}}$ form the <u>Fourier coefficients</u> of f (at infinity).

Note that by uniqueness of Laurent expansions the (a_n) <u>characterize</u> the function f.

It is a fact that <u>much</u> of the arithmetic interest and properties of the modular forms are found in these Fourier coefficients.
In fact, one of the way modular forms can arise in seemingly unrelated problems is by the fact that one may wish to study a sequence $(a_n)_{n\geq 0}$, and discover that the generating series

$$f(z) = \sum_{n\geq 0} a_n\, e(nz)$$

<u>is</u> the Fourier expansion of a modular form.
(When f has level one, this is equivalent to having $f(-\frac{1}{z}) = z^k f(z)$, and therefore reflects a "hidden" symmetry in the coefficients $(a_n)_{n\geq 0}$.)
In this respect, the following result is crucial:
<u>Theorem</u> (cf. Serre, Th. 4 and Cor. 1, Ch. VII.3.2)
For $k \geq 0$, the spaces M_k and M_k^0 are <u>finite-dimensional</u>.

We will see this (with even more precise information) in the next section.

Here we sketch one way this can be used (we will elaborate on this later):

(i) Consider a sequence $(a_n)_{n\geq 0}$ of interest, and assume it is shown that

$$\sum a_n \, e(nz)$$

is in M_k for some (specific) $k \geq 0$.

[Ex. a_n = number of elements with $\|x\|^2 = 2n$ where $x \in E_8 \subset \mathbb{R}^8$ is defined by

$$x = (x_i) \in E_8 \iff \begin{cases} 2x_i \in \mathbb{Z} \\ x_i - x_j \in \mathbb{Z} \\ \sum x_i \in 2\mathbb{Z} \end{cases}$$

(note $x \cdot x \equiv \sum x_i \equiv 0 \ (2)$) with $k = 4$]

(ii) Compute a basis of M_k, say

$$f_1, \dots, f_m$$

[we will see how to do it for k small]

(iii) There exist $c_1, \dots, c_m$ in $\mathbb{C}$ s.t.

$$f = c_1 f_1 + \dots + c_m f_m$$

$\Rightarrow$

$$a_n = c_1 a_n(f_1) + \dots + c_m a_n(f_m)$$

which is an "explicit" formula if the f_i's are explicit.

[Ex. M_4 has dimension 1, is generated by the Eisenstein series E_4

$$\Rightarrow a_n = 240 \sum_{d \mid n} d^3, \text{ by computing the}$$