Chapter II

L-functions and Hecke operators

For a modular form f (of integral weight, but possibly with respect to a subgroup of $SL_2(\mathbb{Z})$, typically $\Gamma_0(q)$ for some $q \geq 1$, and nebentypus χ) we now describe how to associate to f another analytic function ~~which~~, called its (Hecke) L-function. This invariant of f is fundamental to much of the arithmetic aspects of modular forms. In particular:

- it gives a connection with $\zeta(s)$ and Dirichlet L-function
- it is crucial in even stating rigorously the conjectures of Langlands relating modular forms for different groups.

1 – The L-function

Let $f \in S_k(\Gamma_0(q), \chi)$:

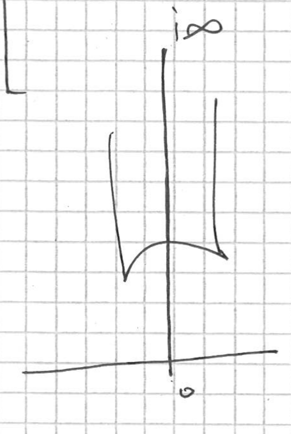
$$f(gz) = \chi(d)\,(cz+d)^k f(z)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $c \equiv 0\,(q)$,

χ a given Dirichlet cha. mod q.

Definition – For $s \in \mathbb{C}$ such that the integral makes sense, one defines

$$\Lambda(f,s) = \int_0^{+\infty} f(iy)\, y^s \,\frac{dy}{y}.$$



Two ~~formal~~ computations reveal some properties of this function, in the case of $f \in S_k(SL_2(\mathbb{Z}))$.

(1) Write $\qquad f(z) = \sum_{n\geq 1} a_n\, e(nz)$

then

$$\Lambda(f,s) \text{ "=" } \sum_{n\geq 1} a_n \underbrace{\int_0^{\infty} e^{-2\pi n y}\, y^s \frac{dy}{y}}_{\frac{1}{(2\pi n)^s}\Gamma(s)}$$

$$= (2\pi)^{-s}\,\Gamma(s) \sum_{n\geq 1} a_n n^{-s}$$

$\rightsquigarrow$ the ~~Dirichlet~~ Hecke Λ-function is ~~related to~~ the Dirichlet generating series of the Fourier coefficients of f.

(2) Since f is modular of weight k, we have

$$f\left(-\frac{1}{z}\right) = z^k f(z)$$

and $\quad -\frac{1}{iy} = \frac{i}{y}$, so $\quad f\left(\frac{i}{y}\right) = (iy)^k f(iy)$

$$\Lambda(f,s) = \int_0^{\infty} f(iy)\, y^{+s}\, \frac{dy}{y}$$

$$= \int_0^{\infty} f\left(\frac{i}{y}\right) (iy)^{-k}\, y^s\, \frac{dy}{y}$$

$$= i^k \int_0^\infty f(iu) \, u^{k-s} \, \frac{du}{u}$$

$u = \frac{1}{y}$, $du = -\frac{dy}{y}$

$$= i^k \Lambda(f, k-s)$$

so we (expect) a functional equation between value of $\Lambda(f,s)$ and $\Lambda(f, k-s)$.

Here is a surprising analogy with properties of Dirichlet L-functions which were known before Hecke:

Th. (Riemann for $\zeta(s)$, ...)

Let $q \geq 1$ be an integer and χ mod q a Dirichlet character mod q. Suppose χ is primitive. Then the function

$$\Lambda(\chi, s) = (q/\pi)^{\frac{s+\delta(\chi)}{2}} \Gamma\left(\frac{s+\delta(\chi)}{2}\right) L(\chi, s)$$

satisfies

$$\Lambda(\bar{\chi}, 1-s) = i^{\delta(\chi)} \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^s \Lambda(\chi, s)$$

where:

(i) $\delta(\chi) = \begin{cases} 0, & \chi(-1) = 1 \\ 1, & \chi(-1) = -1 \end{cases}$

(ii) $\tau(\chi) = \sum \chi(x) \, e\left(\frac{x}{q}\right)$ is a Gauss sum

and χ is primitive means there is no $q' | q$, $q' < q$ and χ' mod q' s.t.

$$\chi(x) = \chi'(x \bmod q') \text{ if } (x, q') = 1$$

(In particular: for $q = 1$, $\chi = 1$,

$$\Lambda(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies

$$\Lambda(1-s) = \Lambda(s)$$

which was proved by Riemann).

This similarity (assuming the computations can be made rigorous] is not accidental at all, and we will in fact prove the Theorem for Dirichlet L-function using modularity properties of $\underline{\Theta\text{-functions}}$, which are modular (of weight $\frac{1}{2}$ however).

But first we must establish rigorously the two results obtained formally.

2 - The Hecke ~~and~~ L and Λ functions

Prop. Let $f \in S_k^{Mk}(q, \chi)$, and let

$$f(z) = \sum_{n \geq 0} a_n\, e(nz)$$

be its Fourier expansion at ∞.

The (a_n) have <u>polynomial growth</u>.

Def. For s s.t. $\mathrm{Re}(s)$ is large enough that it converges one defines

$$L(f,s) = \sum_{n \geq 1} a_n n^{-s}$$

called the (<u>Hecke</u>) L-function of f.

Proof of Prop. - We will in fact obtain a better result

Prop. - We have
$$\sum_{n \leq X} |a_n|^2 \ll X^k$$
where the implied constant depends on f.

Proof. This is quite elegant... Observe that for any fixed $y > 0$ we have
$$f(t+iy) = \sum_{n \geq 1} a_n e^{-2\pi n y} e(nt) \quad \text{for } t \in \mathbb{R}$$
which converges in $L^2([0,1])$, so that
$$\sum_{n \geq 1} |a_n|^2 e^{-4\pi n y} = \int_0^1 |f(t+iy)|^2 \, dt$$
by the classical Parseval relation. Now take $y = \frac{1}{X}$ so that
$$\sum_{n \leq X} |a_n|^2 \leq e^{+4\pi} \sum_{n \geq 1} |a_n|^2 e^{-4\pi n/X}$$
$$\leq e^{4\pi} \int_0^1 |f(t + \tfrac{i}{X})|^2 \, dt$$

We next observe

Lemma. - If $f \in S_k(q, \chi)$ then the function
$$z \longmapsto (\operatorname{Im} z)^{k/2} |f(z)|$$
is bounded on $\mathbb{H}$.

This is because (1) the function is $\Gamma_0(q)$-invariant (elementary check)

(2) $|f(z)| \longrightarrow 0$ exp. fast as z goes to a cusp

So we deduce

$$\left|f\left(t+\frac{i}{x}\right)\right|^2 \ll x^k$$

and the result.

$\square$

<u>Remark</u>. (1) On average, we see that a_n is of size $n^{\frac{k-1}{2}}$ roughly. This ~~is~~ is in fact true individually, up to n^{ε} factors, following from ~~on~~ the work of Deligne.

~~More generally~~ Individually, we get $a_n \ll n^{k/2}$, which is quite good.

(2) The average $\sum_{n\leq x}|a_n|^2 \ll x^k$ is <u>sharp</u>: in fact one can show (Rankin, Selberg) that

$\exists\, c>0$ s.t. $\sum_{n\leq x}|a_n|^2 \sim c\, x^k$, $x\to\infty$.

Maybe unsurprisingly, c is related to the Petersson norm

$$\int_{\Gamma_0(q)\backslash\mathbb{H}} |f(z)|^2\, y^k\, \frac{dx\,dy}{y^2}$$

of f.

<u>Corollary</u> – The Hecke L-function is holomorphic for $\mathrm{Re}(s) > \frac{k}{2}+1$, and

$$\Lambda(f,s) = (2\pi)^{-s}\,\Gamma(s)\,L(f,s)$$

for $\mathrm{Re}(s) > \frac{k}{2}+1$.

<u>Proof</u>. ~~As~~ Apply dominated convergence / monotone convergence

$\square$

Theorem – (Hecke)

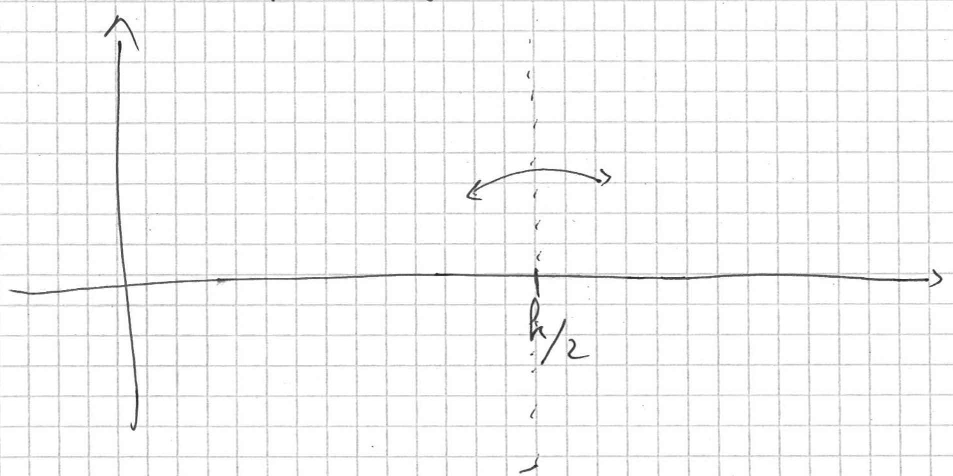
Let $f \in S_k(q,\chi)$

Then $\Lambda(f,s)$ and $L(f,s)$ have hol. continuation to $\mathbb{C}$; more precisely the function

$$\tilde{f}(z) = f|_k \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}(z) = (q^{1/2}z)^{-k} f\left(-\frac{1}{qz}\right)$$

is in $S_k(q,\bar{\chi})$ and for all $s \in \mathbb{C}$, we have

$$q^{s/2}\Lambda(f,s) = i^k \Lambda(\tilde{f}, k-s)\, q^{\frac{k-s}{2}}$$



Proof. The fact that $\tilde{f} \in S_k(q,\bar{\chi})$ is a simple consequence of

Lemma. ~~Suppose~~ $\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} \Gamma_0(q) \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = \Gamma_0(q)$

and χ is mapped to $\bar{\chi}$ by

$$\tilde{\chi}(g) = \chi\left(\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} g \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}\right)$$

Proof - $\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ qc & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ qb & a \end{pmatrix}$

and $\chi(a) = \bar{\chi}(d)$ since $ad \equiv 1 \,(q)$

Now we proceed to the analytic continuation. ~~The key idea is to decompose the integral~~

(2) For analytic continuation we start with the definition

$$\Lambda(f,s) = \int_0^{\infty} f(iy)\, y^{s} \frac{dy}{y}$$

There is no problem, for any $s \in \mathbb{C}$, at ∞, so we split the integral; it is best to split at $\frac{1}{\sqrt{q}}$, as we will see, to get the functional equation.

So

$$\Lambda(f,s) = \int_0^{\frac{1}{\sqrt{q}}} f(iy)\, y^{s} \frac{dy}{y} + \underbrace{\int_{\frac{1}{\sqrt{q}}}^{+\infty} f(iy)\, y^{s} \frac{dy}{y}}_{\text{entire function of } s}$$

To study the first integral, we put $u = \frac{1}{qy}$:

$$\int_0^{\frac{1}{\sqrt{q}}} f(iy)\, y^{s}\, dy/y = \int_{\frac{1}{\sqrt{q}}}^{+\infty} f\left(\frac{i}{qu}\right) (qu)^{-s} \frac{du}{u}$$

Now observe that

$$f\left(\frac{i}{qu}\right) u^{-s} = \tilde{f}(iu)\, q^{k/2}\, i^{k}\, u^{k-s}$$

$$\Big[\text{since } \tilde{f}(iu) = \left(f|_k \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}\right)(iu)$$

$$= q^{k/2} (iqu)^{-k} f\left(-\frac{1}{iqu}\right)$$

$$= q^{-k/2}\, i^{-k} u^{-k} f\left(\frac{i}{qu}\right) \Big]$$

so that

$$\int_0^{\frac{1}{\sqrt{q}}} f(iy)\, y^{s} \frac{dy}{y} = i^{k} q^{\frac{k}{2}-s} \int_{1/\sqrt{q}}^{\infty} \tilde{f}(iu)\, u^{k-s} \frac{du}{u}$$

and further

$$\Lambda(f,s) = \int_{\frac{1}{\sqrt{q}}}^{+\infty} \left(f(iy)\, y^{s} + i^{k} q^{\frac{k}{2}-s} \tilde{f}(iy)\, y^{k-s} \right) \frac{dy}{y}$$

Since $\tilde{f}$ is also a cusp form, this integral exists for all $s \in \mathbb{C}$ and defines an entire function.
But also, if we compare this with the same formula for $\Lambda(\tilde{f},s)$, ~~we get~~ namely

$$\Lambda(\tilde{f},s) = \int_{\frac{1}{\sqrt{q}}}^{+\infty} \left(\tilde{f}(iy)\, y^{s} + i^{k} q^{\frac{k}{2}-s} \tilde{\tilde{f}}(iy)\, y^{k-s} \right) \frac{dy}{y}$$

and observe that $\tilde{\tilde{f}} = f|_k \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}^2 = (-1)^k f$,
we get

$$\Lambda(\tilde{f},s) = \int_{\frac{1}{\sqrt{q}}}^{+\infty} \left(\tilde{f}(iy)\, y^{s} + (-i)^{k} q^{k/2-s} f(iy)\, y^{k-s} \right) \frac{dy}{y}$$

and it follows that

$$q^{s/2} \Lambda(f,s) = i^{k} q^{\frac{k-s}{2}} \Lambda(\tilde{f}, k-s).$$

$\square$

<u>Example.</u>

(1) Consider the Eisenstein series

$$E_k \in M_k(1), \quad k \geq 4.$$

Although E_k is not a cusp form, one can define the Hecke Λ and L-functions in the same way.(*) The analogue of the preceding result is that $L(E_k, s)$ admits analytic continuation to $\mathbb{C}$ but has a <u>pole</u> at $s = k$.

In fact this can be proved directly because E_k has Fourier coefficients proportional to $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and therefore, up to a scalar, $\sum_{n\geq 1} a_n n^{-s} = \sum_{n \geq 1} \left(\sum_{d|n} d^{k-1} \right) n^{-s}$

$$= \left(\sum \frac{1}{n^s} \right) \left(\sum \frac{1}{n^{s-k-1}} \right)$$

(by Dirichlet convolution)

$$= \zeta(s)\, \zeta(s-k+1)$$

This function furthermore has an Euler product, reflecting multiplicativity.

(*) $L(E_k, s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ in terms of Fourier coefficients: the term a_0 does not appear.