Fourier expansion of E_4 and checking the proportionality factor using $a_0 = 1$.]

5 - The dimension of M_k and M_k^0

Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be meromorphic. We denote by $v_z(f)$ the order of f at a point $z \in \mathbb{H}$, ≥ 1 if f has a zero at z, ≤ -1 if it has a pole. If f is modular of some weight k then because $z \longmapsto (cz+d)^k$ has no zero or pole, this order is the same for points in the same $SL_2(\mathbb{Z})$ - orbit. Moreover, we then define $v_\infty(f) = v_0(\tilde{f})$, where

$$f(z) = \tilde{f}(e^{2i\pi z}).$$

($\underline{\underline{\text{Ex.}}}$ $v_\infty(f) = 1$ means that the Fourier expansion starts $a_1 e(z) + a_2 e(2z) + \dots$, $a_1 \neq 0$ and so in particular is a cusp form; in fact

$$M_k^0 = \{ f \in M_k \mid v_\infty(f) \geq 1 \}.)$$

Let
$$\begin{cases} e_z = 1 & \text{if } z \text{ is not equivalent to } i \text{ or } e^{i\pi/3} \\ e_z = 2 & \text{if } z \text{ is equivalent to } i \\ e_z = 3 & \text{if } z \text{ \_\_\_\_\_\_\_\_ } e^{i\pi/3} \end{cases}$$

The key fact is the following formula.

Theorem. (Serre, VII.3 1)

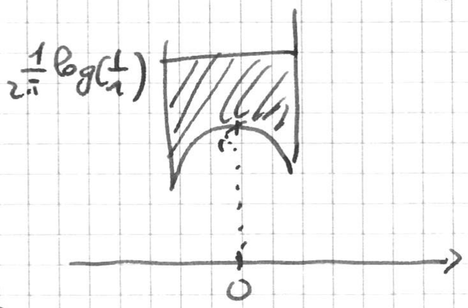
If f is meromorphic including at ∞ and $f \neq 0$ then
$$v_\infty(f) + \sum_{z \in SL_2(\mathbb{Z})\backslash \mathbb{H}} \frac{1}{e_z} v_z(f) = \frac{k}{12}.$$

This constrains in particular the possible number of zeros of $f \in M_k$!

Proof.

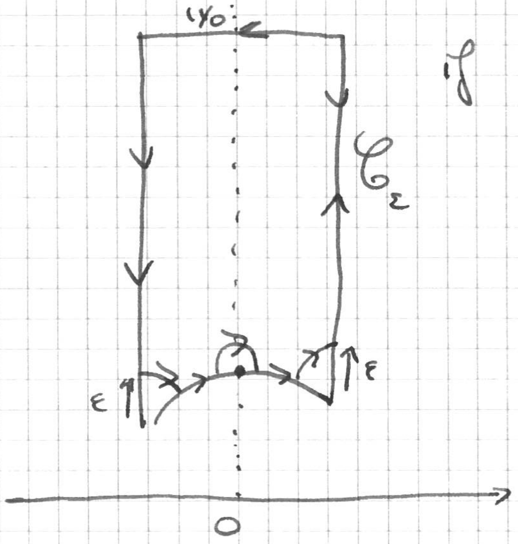
Step 1. The set of z with $v_z(f) \neq 0$ is finite modulo $SL_2(\mathbb{Z})$.

Indeed, since $\tilde{f}$ is meromorphic by assumption, it has no zero and no pole for $0<|w|<r$ for some $r>0$. This corresponds to the region
$$\operatorname{Im} z > \frac{1}{2\pi} \log\left(\frac{1}{r}\right)$$
for f; then since we are looking only at z modulo $SL_2(\mathbb{Z})$, the z with $v_z(f) \neq 0$ and $z \neq \infty$ are represented by some element of
$$\left\{ z \in \mathbb{H} \;\middle|\; |\operatorname{Re} z| \leq \tfrac{1}{2},\ |z| \geqslant 1,\ \operatorname{Im} z \leq \frac{1}{2\pi} \log\left(\frac{1}{r}\right) \right\}$$
which is compact.



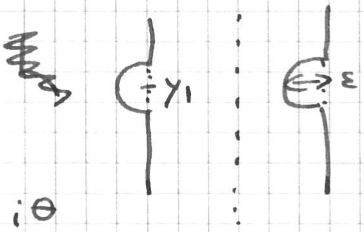
A meromorphic function has finitely many zeros or poles in a compact set, hence the result.

Step 2 – We count the zeros and poles of f, with multiplicity, using Cauchy's Theorem applied to f'/f. Precisely, with r as above chosen so that there is no zero or pole with $\operatorname{Im} z = \frac{1}{2\pi}\log\left(\frac{1}{r}\right) = y_0$, we consider the contour below for $\varepsilon > 0$,

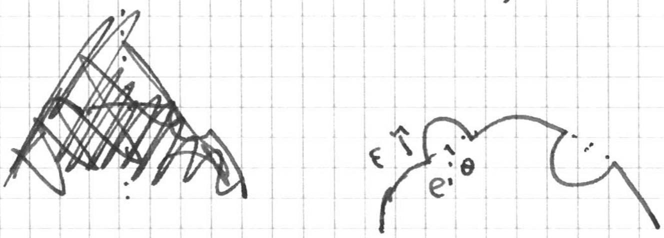


if f has no zero/pole on the boundary.

If there is one at $-\frac{1}{2} + i y_1$, we use



and if there is one at $e^{i\theta}$, we use



where the small arcs are related by $z \longmapsto -\frac{1}{z}$.

Let's assume we are in the case without these possible zeros/poles.

For $\varepsilon > 0$ small enough, we get

$$\frac{1}{2i\pi}\int_{\mathcal{C}_\varepsilon} \frac{f'}{f}\,dz = \sum_{z \in SL_2(\mathbb{Z})\backslash \mathbb{H}}^{*} v_z(f)$$

where the $*$ indicates that we are omitting the orbits of i and of $e^{i\pi/3}$.

Now we transform the integral a bit.

(i) by $z \longmapsto e^{2i\pi z}$, we get

$$\int_{\frac{1}{2}+iy_0}^{-\frac{1}{2}+iy_0} \frac{f'(z)}{f(z)}\,dz = \int_{\circlearrowleft} \frac{\tilde{f}'}{\tilde{f}}(w)\,dw$$

$$= -2\pi\, v_\infty(f)$$

(since $e^{2i\pi(t+iy_0)} = r e^{2i\pi t}$, and t varies from $\frac{1}{2}$ to $-\frac{1}{2}$)

(ii) $\int_{\downarrow} + \int_{\uparrow} = 0$ by periodicity of f

(iii) $\int_{\nearrow} \frac{f'}{f} + \int_{\swarrow} \frac{f'}{f} = \int_{\nearrow} \frac{f'}{f} + \int_{\swarrow} \frac{f'(s.z)}{f(s.z)}\,d(s.z)$

But $f(s.z) = f(-\frac{1}{z})$

$= z^k f(z)$

so $\frac{f'(sz)}{f(sz)} = k\,\frac{1}{z} + \frac{f'(z)}{f(z)}$, and this becomes

$$\int_{\nearrow} \left(-\frac{k}{z}\right) dz \xrightarrow[\varepsilon \to 0]{} \frac{k}{12}$$

(iv) $\int_{i} \longrightarrow \frac{1}{2}\int_{\circlearrowleft i} = -\frac{1}{2}\, v_i(f)$

$\int_{e^{2i\pi/3}} \longrightarrow -\frac{1}{6}\, v_{e^{2i\pi/3}}(f)$, $\int \longrightarrow -\frac{1}{6}\, v_{e^{i\pi/3}}(f)$

All together this gives

$$\lim_{\varepsilon \to 0} \int_{\mathcal{C}_\varepsilon} \frac{f'}{f} = -\frac{1}{3} v_{e(\frac{1}{3})}(f) - \frac{1}{2} v_i(f) - v_\infty(f) + \frac{k}{12}$$

hence the result.

$\square$

We have used the following Lemma repeatedly:

<u>Lemma</u>. Let $f : D \longrightarrow \mathbb{C}$ be meromorphic, $\neq 0$. Let $0 \leq a < b \leq 2\pi$ be real numbers. If $v_0(f) = -1$ then

$$\lim_{\varepsilon \to 0} \int_{\mathcal{C}_{a,b}(\varepsilon)} f(z)\, dz = i(b-a) \operatorname{res}_{z=0} f$$

where

$$\mathcal{C}_{a,b}(\varepsilon) = \left\{ \varepsilon e^{i\theta} \;\middle|\; a \leq \theta \leq b \right\}.$$

<u>Proof</u>. Write $f(z) = \frac{\alpha}{z} + g(z)$ with g hol. in a neighb. of 0. Then

$$\int_{\mathcal{C}_{a,b}(\varepsilon)} f = \int_a^b \frac{\alpha}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\, d\theta + \int_a^b g(\varepsilon e^{i\theta})\, i\varepsilon e^{i\theta}\, d\theta$$

$$= \alpha(b-a) + \underbrace{i\varepsilon \int_a^b g(\varepsilon e^{i\theta}) e^{i\theta}\, d\theta}_{\xrightarrow[\varepsilon \to 0]{} 0}$$

$\square$

In order to fully exploit this formula, we record a first application of the Fourier expansion for Eisenstein series (This will be refined later when computing the Fourier expansion).

Lemma. We have for $k \geq 4$

$$\lim_{y \to \infty} E_k(z) = 1$$

and in particular $E_k \in M_k - M_k^0$.

Proof. Recall that

$$E_k(z) = \sum_{g \in H \backslash SL_2(\mathbb{Z})} \frac{1}{(cz+d)^k}$$

where $H = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$.

If $g \neq \pm Id \mod H$ ($\Leftrightarrow c \neq 0$) then

$$|cz+d|^2 = (cx+d)^2 + c^2y^2 \geq c^2y^2$$

so $\dfrac{1}{(cz+d)^k} \longrightarrow 0$ as $y \to \infty$. In fact

$$|cz+d|^2 = c^2x^2 + d^2 - 2cd + c^2y^2$$
$$\geq c^2 - cd + d^2 = |ce(\tfrac{1}{3}) - d|^2$$

which shows the convergence is dominated, hence

$$\lim \sum_{g \in H \backslash SL_2(\mathbb{Z})} \frac{1}{(cz+d)^k} = \sum_{g \in H \backslash SL_2(\mathbb{Z})} \lim_{y \to \infty} \frac{1}{(cz+d)^k}$$

$$= \lim_{y \to \infty} 1 = 1.$$

from $g = \pm Id$

From this, it follows that the function $\tilde{f}$ for E_k is <u>bounded</u> in a neighborhood of 0. Since it is meromorphic, it follows that it is in fact holomorphic at 0, so that $E_k \in M_k$. And since it takes the value 1, it is not in $\overset{\circ}{M}_k$.

$\square$

<u>Note</u>. We are using an analytic normalization of E_k, which explains the discrepancy with Prop. 4 in Serre $\underline{\text{VII}}$. 2.3. This will be further clarified when we compute later the full Fourier expansion of Eisenstein / Poincaré series.

<u>Theorem</u>. (1) $M_k = \{0\}$ for $k < 0$ and k odd and moreover for $\ell \geq 1$ integer

$$\dim M_{2\ell} = \begin{cases} \lfloor \ell/6 \rfloor & \text{if } \ell \equiv 1 \ (6) \\ \lfloor \ell/6 \rfloor + 1 & \text{if } \ell \not\equiv 1 \ (6) \end{cases}$$

(2) E_4, E_6, E_8, E_{10} are bases of the one-dimensional spaces M_4, M_6, M_8, M_{10}.

(3) $\overset{\circ}{M}_k = \{0\}$ for $k < 12$ or $k = 14$, and otherwise $\dim \overset{\circ}{M}_k = \dim M_k - 1$ with $E_k \in M_k - \overset{\circ}{M}_k$.

(4) $\Delta = E_4^3 - E_6^2$ is a basis of $\overset{\circ}{M}_{12}$.

(5) $f \longmapsto f\Delta$ is an isomorphism $M_{k-12} \longrightarrow \overset{\circ}{M}_k$.