Proof. (1) For $f \in M_{2k}$, $f \neq 0$, all $v_z(f)$ are ≥ 0 integers, so the formula above becomes

$$(*) \qquad v_\infty(f) + \frac{v_i(f)}{2} + \frac{v_{e(1/3)}(f)}{3} + \Big(\sum_{\substack{\text{other}\\ \text{pts}}} v_z(f)\Big) = \frac{2k}{12}.$$

In particular, $k < 0$ is impossible, as well as $k = 1$ (i.e. $M_2 = \{0\}$) because $\frac{1}{6}$ can not be written $a + \frac{b}{2} + \frac{c}{3}$, $a, b, c \geq 0$ integers. So $M_{2k} = \{0\}$ if $k < 0$ or $k = 1$.

(2) Claim: there exists a function $\Delta \in M_{12}^0 - \{0\}$ such that $v_z(\Delta) = 0$ for $z \in \mathbb{H}$.

Indeed, let

$$\Delta = E_4^3 - E_6^2$$

Then $\Delta \in M_{12}$ and its value at ∞ is $1 - 1 = 0$, according to the previous lemma. We need to check that $\Delta \neq 0$. For this we apply $(*)$ to $f = E_4$ and E_6; the only possibilities are

$$v_i(E_4) = 0, \quad v_{e(1/3)}(E_4) = 1$$

$$v_i(E_6) = 1, \quad v_{e(1/3)}(E_4) = 0$$

(and all other values are 0), so $\Delta(i) \neq 0$ for instance.

(3) Claim: $\dim M_k^0 = \dim M_k$ or $\dim M_k - 1$

Indeed, $M_k^0 = \ker(f \longmapsto \tilde{f}(0))$, the kernel of a (possibly zero) linear form.

(4) Claim: $f \overset{u}{\longmapsto} f\Delta$ is an isomorphism of $\mathbb{C}$-vector spaces from M_{2k} to M^0_{2k+12}.

Indeed, this map u is linear and injective, and $f\Delta(\infty) = 0$, so it does have image in M^0_{2k+12}. We need then only to check that it is surjective: let $f \in M^0_{2k+12}$; define $g = \frac{f}{\Delta}$; then g is modular of weight $2k$, and in fact is holomorphic everywhere since $v_\infty(g) = v_\infty(f) - 1 \geq 0$

$z \in \mathbb{H} \Rightarrow v_z(g) = v_z(f) \geq 0$.

Since $u(g) = f$, we have the result.

(5) Combining (1) and (4) we have the conclusion that $M^0_{2k} = \{0\}$ for $2k \leq 10$ (or $2k = 14$), so $M_0, M_4, M_6, M_8, M_{10}$ must be generated by the non-zero elements $1, E_4, E_6, E_8, E_{10}$. (So we see that it was not a coincidence that E_2 does not converge...). Then $M^0_{12} \simeq M_0$ is one-dimensional, so generated by Δ.

(6) Finally observe that the RHS and LHS of the formula in statement (1) coincide for $2k \leq 12$,

and increase by 1 when replacing k by $k+6$.
So the formula holds for <u>all</u> $k \geq 0$ by induction.

$\square$

Corollary. - (1) The $\mathbb{C}$-algebra

$$\bigoplus_{k\geq 0} M_k$$

(with product of modular forms as multiplication) is isomorphic to $\mathbb{C}[X, Y]$ by $\begin{cases} X \mapsto E_4 \\ Y \mapsto E_6 \end{cases}$

(2) There exists a meromorphic modular form $j : \mathbb{H} \longrightarrow \mathbb{C}$ of weight 0 such that j defines a bijection

$$\mathbb{H} / SL_2(\mathbb{Z}) \longrightarrow \mathbb{C}$$

and j has a pole at ∞ with residue 1.

Proof. (1) Since $\begin{cases} E_4(e^{2i\pi/3}) = 0, & E_6(e^{2i\pi/3}) \neq 0 \\ E_6(i) = 0, & E_4(i) \neq 0 \end{cases}$

the obvious map

$$\begin{array}{ccc} \mathbb{C}[X, Y] & \xrightarrow{u} & \bigoplus M_k \\ X & \longmapsto & E_4 \\ Y & \longmapsto & E_6 \end{array}$$

is injective: indeed, otherwise we would have for some k even a relation

$$\sum_{4i+6j=k} \alpha_{ij} E_4^i E_6^j = 0 \quad , \text{ not all } \alpha_{ij} \text{ zero}$$

Picking a, b with $4a+6b=k$ and a minimal, we observe that for all i, j with $4i+6j=k$, we have $i \geq a$ so $j \leq b$, and

$$E_4^i E_6^j = E_4^a E_6^b \frac{E_6^{j-b}}{E_4^{i-a}}$$

and this is $E_4^a E_6^b f^{\frac{j-b}{2}}$ for

$$f = \frac{E_6^2}{E_4^3}$$

(since $4(a-i) = 6(j-b)$, we have $2 \mid j-b$ and $3 \mid a-i$). So f would solve an algebraic equation with constant coefficients, which is only possible if f is constant, which the values at i, $e(1/3)$ show isn't the case.

To prove that u is <u>surjective</u>, we argue by induction on k that $M_k \subset \mathrm{Im}(u)$. This holds for $k \leq 6$ since $M_0 = \mathbb{C}\cdot 1$, $M_2 = \{0\}$, $M_4 = \mathbb{C}\cdot E_4$, $M_6 = \mathbb{C}E_6$. Otherwise, pick a, b with

$$4a + 6b = k$$

(Exercise: all even $k \geq 4$ are of this form) and note that if $f \in M_k$ then there is λ s.t. $f - \lambda E_4^a E_6^b \in M_k^0$ (because $E_4^a E_6^b(\infty) \neq 0$), hence

$$f = \lambda E_4^a E_6^b + \Delta g$$

for some $g \in M_{k-12}$, and we conclude the induction since $\Delta \in \mathrm{Im}(u)$ by definition.

(2) Since Δ has a simple zero at ∞ and no other zero, we can find $\lambda \in \mathbb{C}^\times$ such that $j = \lambda \dfrac{E_4^3}{\Delta}$ has a simple pole with residue 1 at ∞. It is of weight zero, and holomorphic on $\mathbb{H}$. (But $j \notin M_0$ since it is not holomorphic at ∞.)

So j defines certainly a ~~holo~~ map

$$SL_2(\mathbb{Z}) \backslash \mathbb{H} \longrightarrow \mathbb{C}$$

To prove this is a bijection, pick $z \in \mathbb{C}$. The function

$$f_z = \lambda E_4^3 - z\Delta$$

is in M_{12}, and is $\neq 0$ (since Δ, E_4^3 differ at ∞). The formula of p° 28 gives

$$0 = v_\infty(f_z) + \frac{1}{2} v_i(f_z) + \frac{1}{3} v_{e(1/3)}(f_z) + \sum_{\substack{\text{other} \\ w}} v_w(f_z)$$

$$= 1$$

The only possible choices are

$v_i(f_z) = 2$, others are 0

$v_{e(1/3)}(f_z) = 3$, —— 0

$\exists$ unique $w \neq i, e(1/3)$ s.t. $v_w(f_z) = 1$, others are 0.

In any case, f_z has a <u>unique</u> zero $w \in \mathbb{H} \backslash SL_2(\mathbb{Z})$
so a <u>unique</u> solution to

$$j(w) = z.$$

$\square$