$k \geq 0$ even

Lemma. For $f \in M_k^\circ$, the integral
$$\int_D |f(z)|^2 \, y^k \, \frac{dx\,dy}{y^2}$$
is finite, where
$$D = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1, \; |\mathrm{Re}(z)| \leq \tfrac{1}{2} \right\}.$$

Proof. Since $f \in M_k^\circ$, there is a Fourier expansion
$$f(z) = \sum_{n \geq 1} a_n \, e(nz)$$
and hence for $\mathrm{Im}(z)$ large enough we get
$$|f(z)| \leq 2|a_1| \, |e(z)| = 2|a_1| \, e^{-2\pi y}$$
and so the function $(x,y) \longmapsto |f(z)|^2 \, y^{k-2}$ is Lebesgue-integrable in D.

$\square$

Definition. The Petersson inner-product on M_k° is defined by
$$\langle f, g \rangle = \int_D f(z) \overline{g(z)} \, y^k \, \frac{dx\,dy}{y^2}.$$

This makes M_k° a (finite-dimensional) Hilbert space, since $\langle f, f \rangle = 0$, then $f = 0$ on D, hence $f = 0$.

In a Hilbert space E, we know that any continuous linear form is of the form $x \longmapsto \langle x, y \rangle$ for some $y \in E$.

Theorem (Petersson). For $k \geq 4$, $m \geq 1$, $f \in M_k^\circ$, we have
$$\langle f, P_{m,k} \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \, a_m(f)$$
where
$$f(z) = \sum_{n \geq 1} a_n(f) \, e(nz)$$
is the Fourier expansion of f.

Corollary. The Poincaré series $P_{m,k}$ for $m \geq 1$ generate M_k° as $\mathbb{C}$-vector space.

Proof. If $f \in M_k^\circ$ is orthogonal to all Poincaré series, then by the Theorem we get $a_m(f) = 0$ for all $m \geq 1$, hence $f = 0$. So the space generated by $\{P_{m,k} \mid m \geq 1\}$ must be M_k°.

$\square$

Proof of Theorem. ("Unfolding technique")

$\Gamma = SL_2(\mathbb{Z})$
$\overline{N} = \{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \}$

We compute

$$\langle f, P_{m,k} \rangle = \int_D f(z) \overline{P_{m,k}(z)} \, y^k \frac{dx\,dy}{y^2}$$

$$= \int_D f(z) \overline{\sum_{g \in \overline{N} \backslash \Gamma} \frac{e(mgz)}{(cz+d)^k}} \, y^k \frac{dx\,dy}{y^2}$$

$$= \int_D \sum_{g \in \overline{N} \backslash \Gamma} \frac{f(gz)}{(cz+d)^k} \overline{\frac{e(mgz)}{(cz+d)^k}} \, y^k \frac{dx\,dy}{y^2}$$

$\left(\mathrm{Im}(gz) = \frac{y}{|cz+d|^2} \right)$

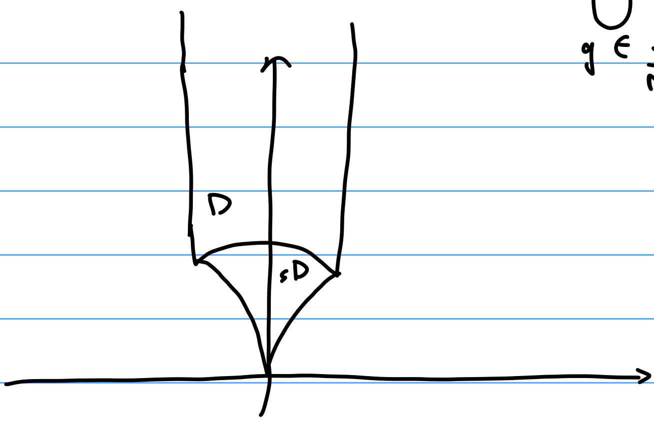
$$= \int_D \sum_{g \in \overline{N} \backslash \Gamma} f(gz) \overline{e(mgz)} \, \mathrm{Im}(gz)^k \frac{dx\,dy}{y^2}$$

$$= \sum_{g \in \overline{N} \backslash \Gamma} \int_D f(gz) \overline{e(mgz)} \, \mathrm{Im}(gz)^k \frac{dx\,dy}{y^2}$$

$$= \sum_{g \in \overline{N} \backslash \Gamma} \int_{gD} f(w) \overline{e(mw)} \, \mathrm{Im}(w)^k \frac{du\,dv}{v^2}$$

$(w = u + iv)$

$$= \int_{\bigcup_{g \in \overline{N} \backslash \Gamma} gD} f(w) \overline{e(mw)} \, \mathrm{Im}(w)^k \frac{du\,dv}{v^2}$$



$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\bigcup_{g \in \overline{N} \backslash \Gamma} gD$ is a fundamental domain for the action of $\bar{N}$ on $\mathbb{H}$ (i.e. every element in $\mathbb{H}$ is in this modulo $\bar{N}$, with "no" or almost no repetition).

This means that $\displaystyle\bigcup_{g \in \bar{N}\backslash\Gamma} gD$ " $=$ " $\{ z \in \mathbb{H} \mid |\mathrm{Re}\, z| \leq \frac{1}{2} \}$

so

$$< f, P_{m,k} > = \int_{-1/2}^{\frac{1}{2}} \int_0^{+\infty} f(u+iv)\, \overline{e(m(u+iv))}\, v^{k-2}\, du\, dv$$

$$= \int_0^{+\infty} v^{k-2} \left(\int_{-1/2}^{1/2} f(u+iv)\, \overline{e(m(u+iv))}\, du \right) dv$$

Note $\overline{e(m(u+iv))} = \overline{\exp(2i\pi mu - 2\pi mv)}$

$$= \exp(-2\pi m v) \exp(-2i\pi mu)$$

so

$$< f, P_{m,k} > = \int_0^{\infty} v^{k-2} e^{-2\pi mv} \left(\int_{-1/2}^{1/2} f(u+iv)\, e(-mu)\, du \right) dv$$

Now plug the Fourier expansion

$$f(u+iv) = \sum_{n \geq 1} a_n(f)\, e^{-2\pi n v}\, e(nu)$$

to get

$$< f, P_{m,k} > = \int_0^{\infty} v^{k-2} e^{-2\pi(m+n)v} \left(\sum_{n\geq 1} a_n(f) \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e((n-m)u)\, du}_{= \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}} \right) dv$$

$$= a_m(f) \int_0^{+\infty} v^{k-2} e^{-4\pi m v}\, dv$$

$$= a_m(f) \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \quad (\text{by def. of } \Gamma).$$

$\square$

(Note - check that the computation is justified.)

Corollary - ("The Petersson formula")

Let $k \geq 4$ integer. Let $\mathcal{F}$ be an orthonormal basis of M_k^0. Then for $m \geq 1$, $n \geq 1$

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f\in\mathcal{F}} a_f(m)\,\overline{a_f(n)} = \delta(m,n) + \frac{2\pi}{i^k}\sum_{c\geq 1}\frac{1}{c} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

Proof. We can expand $P_{m,k}$ in the basis $\mathcal{F}$:

$$P_{m,k} = \sum_{f\in\mathcal{F}} \langle P_{m,k}, f\rangle f$$
$$= \sum_{f\in\mathcal{F}} \overline{\langle f, P_{m,k}\rangle} f$$
$$\overset{\text{(Theorem)}}{=} \sum_{f\in\mathcal{F}} \overline{a_f(m)} \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} f$$

which gives for n-th Fourier coefficients

$$\delta(m,n) + \frac{2\pi}{i^k}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\sum_{c\geq 1}\frac{1}{c} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{f\in\mathcal{F}} \overline{a_f(m)}\, a_n(f) \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}$$

hence the result.

$\square$

Note. This result is a type of "quasi-orthogonality" property for the Fourier coefficients of cusp forms $\longrightarrow$ "relative trace formulas".

Example. $k = 12$

There is the generator Δ of M_{12}^0 given by

$$\Delta(z) = e(z)\prod_{n\geq 1}(1-e(nz))^{24}$$
$$= \sum_{n\geq 1}\tau(n)\, e(nz)$$

where $\tau(n)$ is an integer for all $n\geq 1$ ("Ramanujan tau-function")

Let $m\geq 1$. Then there exists $c(m)\in\mathbb{C}$ such that

$$P_{m,12} = c(m)\,\Delta.$$

Then

$$\underset{\displaystyle \frac{\Gamma(11)}{(4\pi m)^{11}}\tau(m)}{\underset{\|}{\langle \Delta, P_{m,12}\rangle}} = c(m)\,\|\Delta\|^2$$

so

$$P_{m,12} = \frac{10!}{(4\pi m)^{10}}\tau(m)\,\Delta.$$

Apply the Petersson formula to M_{12}° with $\mathcal{F} = \left\{ \frac{\Delta}{\|\Delta\|} \right\}$ to get

$$\frac{10!}{(4\pi\sqrt{mn})^{11}} \frac{\tau(m)\tau(n)}{\|\Delta\|^2} = \delta(m,n) + 2\pi \sum_{c\geq 1} \frac{1}{c} S(m,n;c) J_{11}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

For $m = 1$, $n \geq 2$: $\tau(1) = 1$ by definition

so

$$\frac{10!}{(4\pi\sqrt{n})^{11}} \frac{\tau(n)}{\|\Delta\|^2} = 2\pi \sum_{c\geq 1} \frac{1}{c} S(m,n;c) J_{11}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

Note that we see a factor $n^{11/2}$ come out naturally.
Deligne's Theorem (conj. of Ramanujan) essentially means that

$$|\tau(n)| = O\left(n^{11/2+\varepsilon}\right) \quad \text{for every } \varepsilon > 0.$$

(Nobody has managed to prove this using this formula!)

Lemma. If $c \geq 1$ and $m \geq 1$, $n \geq 1$ are coprime then

$$S(m,n;c) = S(mn,1;c).$$

Assuming this, we get for m and n coprime

$$\frac{10!}{(4\pi\sqrt{mn})^{11}} \tau(m)\tau(n) = 2\pi \sum_{c\geq 1} \frac{1}{c} S(mn,1;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

$$= \frac{10!}{(4\pi\sqrt{mn})^{11}} \tau(mn)$$

so $\boxed{\text{the tau-function is multiplicative!}}$ (Ramanujan; Mordell; Hecke)

Proof of Lemma.

First if m is coprime to c then

$$S(m,n;c) = \sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{md + n\bar{d}}{c}\right), \qquad d\bar{d} \equiv 1 \ (c)$$

$$= \sum_{\substack{x \bmod c \\ (x,d)=1}} e\left(\frac{x + mn\bar{x}}{c}\right) \qquad \left(\begin{array}{l} x = md \\ d = \bar{m}x \end{array}\right.$$

$$= S(1,mn;c) = S(mn,1;c)$$

so the formula is easy in that case.

General case [recent elementary argument of Xi Ping]:

$$S(m,n;c) = \sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{md+n\bar{d}}{c}\right)$$

$$= \sum_{\substack{x,y \bmod c \\ xy=1}} e\left(\frac{mx+ny}{c}\right)$$

$$= \sum_{x,y \bmod c} e\left(\frac{mx+ny}{c}\right) \frac{1}{c}\sum_{a \bmod c} e\left(\frac{a(xy-1)}{c}\right)$$

$$= \frac{1}{c}\sum_{a \bmod c} e\left(-\frac{a}{c}\right) \underbrace{\sum_{x,y} e\left(\frac{mx+ny+axy}{c}\right)}_{T(a;c)}$$

<u>Claim</u>: the inner sum $\sum_{x,y} e\left(\frac{mx+ny+axy}{c}\right)$ is zero unless a is coprime with c

(Indeed, note that summing over x first we get

$$T(a;c) = \sum_{y(c)} e\left(\frac{ny}{c}\right) \sum_{x(c)} e\left(\frac{x(m+ay)}{c}\right)$$

$$= c\sum_{y(c)} e\left(\frac{ny}{c}\right) \mathbb{1}(m+ay=0 \ (c))$$

so this is zero unless there is some y such that

$$m+ay=0 \quad (c).$$

Similarly, we find that $T(a;c)=0$ unless there is some x such that

$$n+ax=0 \quad (c).$$

If there is a $p \mid (a,c)$ one gets $\begin{cases} m=0 & (p), \\ n=0 & (p) \end{cases}$

which is impossible if $(m,n)=1$.)

So

$$S(m,n;c) = \frac{1}{c} \sum_{\substack{a \bmod c \\ (a,c)=1}} e\left(-\frac{a}{c}\right) \underbrace{T(a;c)}_{c\, e\left(\frac{ny}{c}\right)}$$ where

$m + ay \equiv 0\ (c)$

$\iff y = -m\bar{a}$

$$\Rightarrow S(m,n;c) = \sum_{\substack{a \bmod c \\ (a,c)=1}} e\left(-\frac{a}{c} - \frac{mn\bar{a}}{c}\right)$$

$$= S(-1, -mn; c)$$

$$= S(1, mn; c)$$

$$= S(mn, 1; c).$$

$\square$