(Reference : Iwaniec, "Topics in classical automorphic forms" Section 3.2)

Theorem. Let $\begin{cases} k \geq 4 & \text{integer} \\ m \geq 0 & \text{integer} \end{cases}$

(1) The Fourier expansion of $E_k = P_{0,k}$ is given by

$$E_k(z) = 1 + \frac{(2i\pi)^k}{\zeta(k)(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) e(nz) \qquad [e(z) = e^{2i\pi z}]$$

where $$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

(2) For $m \geq 1$, the Fourier expansion of $P_{m,k}$ is given by

$$P_{m,k}(z) = \sum_{n \geq 1} p_k(m,n) e(nz)$$

where

$$p_k(m,n) = \underbrace{\delta(m,n)}_{= \begin{cases} 1, & m=n \\ 0, & m \neq n \end{cases}} + \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \frac{2\pi}{i^k} \sum_{c \geq 1} \frac{1}{c} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where

$$S(m,n,c) = \sum_{\substack{x \bmod c \\ (x,c)=1}} e\left(\frac{mx + n\bar{x}}{c}\right), \qquad x\bar{x} \equiv 1 \bmod c$$

("Kloosterman sum")

$$J_\nu(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+\nu)} \left(\frac{z}{2}\right)^{\nu+2n} \qquad \left(\begin{array}{l} \nu > 0 \\ z \in \mathbb{C} \end{array}\right)$$

("Bessel function")

Remarks. (1) We see that $P_{m,k} \in M_k^0$ if $m \geq 1$; but they may vanish (in fact, $P_{m,4} = P_{m,6} = \cdots = P_{m,10} = 0$ for all $m \geq 1$). In general, there are many linear relations satisfied by the Poincaré series.

(2) One can show that the $P_{m,k}$ generate M_k^0 (see later).

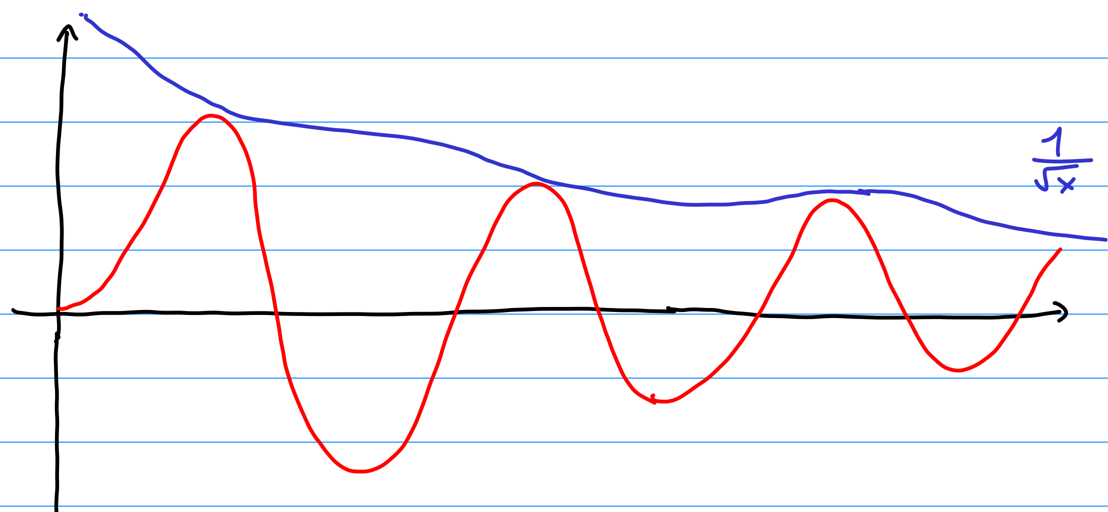
(3) The series defining $p_k(m,n)$ converges absolutely: note that $|S(m,n;c)| \leq c$ and that in $J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$, as $c \to \infty$, we look at $J_{k-1}(z)$ for z close to 0, which satisfies $J_{k-1}(z) = O(z^{k-1})$ for $|z| \leq 1$, hence

$$\frac{1}{c} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) = O\left(\frac{1}{c} \cdot c \cdot \frac{1}{c^{k-1}}\right)$$

(for fixed m and n), which defines a convergent series.

(4) This formula was found by Poincaré (~ 1910), and rediscovered by Petersson ($\sim 1920/30$).

(5) The graph of $J_{k-1}(x)$ for $x \geq 0$ has the following shape:



(Oscillations like cos or sin, decay like $\frac{1}{\sqrt{x}}$)

(<u>Much</u> is known about Bessel functions.)

Proof. Ingredients: recall that

$$P_{m,k}(z) = \sum_{g \in \{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix} \mid n\in\mathbb{Z}\}\backslash SL_2(\mathbb{Z})} \frac{e(mgz)}{(cz+d)^k}.$$

① Parameterization of $\{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix} \mid n\in\mathbb{Z}\}\backslash SL_2(\mathbb{Z})$ ("Bruhat decomposition")

② Poisson summation formula: for suitable $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ we have

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{h\in\mathbb{Z}} \underbrace{\hat{\varphi}(h)}_{\text{Fourier transform}}$$

③ Identities for special functions

Lemma 1. Let $\Gamma = SL_2(\mathbb{Z})$

$$\bar{N} = \left\{ \pm\begin{pmatrix}1&n\\0&1\end{pmatrix} \mid n\in\mathbb{Z}\right\}$$

$$\cup$$

$$N = \left\{ \begin{pmatrix}1&n\\0&1\end{pmatrix} \mid n\in\mathbb{Z}\right\}$$

Then we have a disjoint union

$$SL_2(\mathbb{Z}) = \bar{N} \cup \bigcup_{c\geq 1} \bigcup_{\substack{0\leq d\leq c-1\\(c,d)=1}} \bar{N}\begin{pmatrix}* & *\\ c & d\end{pmatrix}N$$

where $\begin{pmatrix}* & *\\ c & d\end{pmatrix}$ is any fixed element of $SL_2(\mathbb{Z})$ with $(c\ d)$ as bottom row.

Proof. Let $g\in\Gamma$, $g = \begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$, $\alpha\delta-\beta\gamma = 1$

If $\gamma = 0$, then $g\in\bar{N}$.

Suppose $\gamma \neq 0$. First, assume $\gamma\geq 1$. Note that

$$\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\begin{pmatrix}1&n\\0&1\end{pmatrix} = \begin{pmatrix}\alpha&\alpha n+\beta\\\gamma&\gamma n+\delta\end{pmatrix}$$

so we see that there is a unique $n \in \mathbb{Z}$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ c & d \end{pmatrix}$$

with $\begin{cases} c = \gamma \\ 0 \leq d \leq c-1 \end{cases}$ (and $(c,d) = 1$ since $\alpha' d - \beta' c = 1$)

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the fixed matrix in $Sl_2(\mathbb{Z})$ with bottom row (c,d) chosen in the union above. Using

$$\alpha' d - \beta' c = ad - bc = 1$$

we see that there is a unique $m \in \mathbb{Z}$ such that

$$\begin{cases} \alpha' = a + mc \\ \beta' = b + md \end{cases}$$

so that

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ c & d \end{pmatrix} = g \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$

$$g = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \in N \begin{pmatrix} a & b \\ c & d \end{pmatrix} N.$$

If $\gamma \leq -1$, then we do the same for $-g$, so

$$-g \in N \begin{pmatrix} a & b \\ c & d \end{pmatrix} N \Rightarrow g \in \bar{N} \begin{pmatrix} a & b \\ c & d \end{pmatrix} N.$$

To prove finally that the union is <u>disjoint</u>, we just look at the argument above: the integers m, n are unique.

$\square$

<u>Corollary</u> - Let $\varphi : \mathbb{H} \longrightarrow \mathbb{C}$ be any function such that $\sum_{g \in \bar{N} \backslash \Gamma} (\varphi|_k g)(z)$ <u>absolutely</u> converges, then

$$\sum_{g \in \bar{N} \backslash \Gamma} (\varphi|_k g)(z) = \varphi(z) + \sum_{c \geq 1} \sum_{\substack{1 \leq d \leq c-1 \\ (c,d)=1}} \sum_{m \in \mathbb{Z}} \frac{\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} z\right)}{(c(z+m)+d)^k}.$$

[Poisson summation formula]

Lemma 2. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a C^1 function such that integrable

$$\sum_{n\in\mathbb{Z}} \varphi(n+x)$$
$$\sum_{n\in\mathbb{Z}} \varphi'(n+x)$$

converge locally uniformly for $x \in \mathbb{R}$.

Then
$$\sum_{n\in\mathbb{Z}} \varphi(n) = \sum_{h\in\mathbb{Z}} \hat{\varphi}(h)$$
where
$$\hat{\varphi}(t) = \int_{\mathbb{R}} \varphi(x)\, e(-xt)\, dx$$
is the Fourier transform of φ.

[Note: $\hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x) dx$]

Proof. We define $f : \mathbb{R} \longrightarrow \mathbb{C}$ by
$$f(x) = \sum_{n\in\mathbb{Z}} \varphi(n+x),$$
so the assumptions imply that $f \in C^1(\mathbb{R})$; moreover f is periodic of period 1 by averaging.

So f is the sum of its Fourier series: for $x \in \mathbb{R}$, we have
$$f(x) = \sum_{h\in\mathbb{Z}} \hat{f}(h)\, e(hx)$$
(and in particular
$$\sum_{n\in\mathbb{Z}} \varphi(n) = f(0) = \sum_{h\in\mathbb{Z}} \hat{f}(h) \quad)$$
where
$$\hat{f}(h) = \int_0^1 f(x)\, e(-xh)\, dx.$$

We claim to conclude that $\hat{f}(h) = \hat{\varphi}(h)$ for $h \in \mathbb{Z}$.

To see, we compute:
$$\hat{f}(h) = \int_0^1 \Big(\sum_{n\in\mathbb{Z}} \varphi(n+x)\Big)\, e(-xh)\, dx$$
$$\overset{①}{=} \sum_{n\in\mathbb{Z}} \int_0^1 \varphi(n+x)\, e(-xh)\, dx$$
$$\overset{②}{=} \sum_{n\in\mathbb{Z}} \int_n^{n+1} \varphi(x)\, e(-xh)\, dx \qquad [e(nh)=1]$$
$$\overset{③}{=} \int_{-\infty}^{+\infty} \varphi(x)\, e(-xh)\, dx = \hat{\varphi}(h)$$

where ① holds by local uniform convergence of f
② linear change of variable
③ countable additivity of Lebesgue measure.

$\square$

Recall :
$$P_{m,k}(z) = \sum_{g \in \Gamma_{\overline{N}} \backslash \Gamma} \frac{e(mgz)}{(cz+d)^k}.$$

We use the Corollary of Lemma 1 to write
$$P_{m,k}(z) = e(mz) + \sum_{c \geq 1} \sum_{\substack{(c,d)=1 \\ 0 \leq d \leq c-1}} \sum_{n \in \mathbb{Z}} \frac{e\left(m \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z+n)\right)}{(c(z+n)+d)^k}.$$

Note that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z+n) = \frac{a(z+n)+b}{c(z+n)+d}$$
$$= \frac{a}{c} - \frac{1}{c(cz+cn+d)} \qquad (ad-bc=1)$$

so the series is
$$e\left(\frac{am}{c}\right) \sum_{c \geq 1} \sum_{\substack{0 \leq d \leq c-1 \\ (c,d)=1}} \sum_{n \in \mathbb{Z}} (cz+cn+d)^{-k} e\left(-\frac{m}{c(cz+cn+d)}\right).$$

For fixed c, d, we apply the Poisson Summation Formula to
$$\varphi(x) = \frac{1}{(cz+cx+d)^k} e\left(-\frac{m}{c(cz+cx+d)}\right).$$

We check the assumptions:

(i) $\varphi \in C^\infty(\mathbb{R})$, $|e(-\dots)| \leq 1$, $k \geq 4$ so $\varphi \in L^1(\mathbb{R})$

(ii) similarly, $\sum_{n \in \mathbb{Z}} \varphi(n+x)$ is locally uniformly absolutely convergent

(iii) $$\varphi'(x) = -kc(cz+cx+d)^{-k-1} e(\underline{\quad}) + (cz+cx+d)^{-k} \times \frac{-2i\pi m}{c} (cz+cx+d)^{-2} \cdot e(\underline{\quad})$$

which decays like $\frac{1}{x^{k+1}}$ at ∞, so

$$\sum \varphi'(n+x)$$

is also loc. unif.^{ly} abs. convergent.

Lemma 2 $\Rightarrow$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(cz+cn+d)^k}\, e\left(-\frac{m}{c(cz+cn+d)}\right) = \sum_{h\in\mathbb{Z}} \hat{\varphi}(h)$$

where

$$\hat{\varphi}(h) = \int_{\mathbb{R}} \frac{1}{(cz+ct+d)^k}\, e\left(-\frac{m}{c(cz+ct+d)} - ht\right) dt.$$

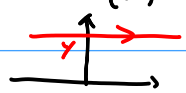
Write $z = x+iy$; put $u = t + x + \frac{d}{c}$, then

$$\hat{\varphi}(h) = \int_{\mathbb{R}} \frac{1}{(cu+icy)^k}\, e\left(-\frac{m}{c^2(u+iy)} - h\left(u - x - \frac{d}{c}\right)\right) du$$

$$= e\left(\frac{hd}{c}\right) e(hx) \int_{\mathbb{R}} \frac{1}{(cu+icy)^k}\, e\left(-\frac{m}{c^2(u+iy)} - hu\right) du$$

$$= e\left(\frac{hd}{c}\right) e(hz) \int_{\mathbb{R}} \frac{1}{(c(u+iy))^k}\, e\left(-\frac{m}{c^2(u+iy)} - h(u+iy)\right) du$$

$$= e\left(\frac{hd}{c}\right) e(hz) \int_{\operatorname{Im}(w)=y} \frac{1}{(cw)^k}\, e\left(-\frac{m}{c^2 w} - hw\right) dw$$



Summary: for $z \in \mathbb{H}$

$$P_{m,k}(z) = e(mz) + \sum_{c\geq 1} \sum_{\substack{0\leq d\leq c-1 \\ (c,d)=1}} \sum_{h\in\mathbb{Z}} e\left(\frac{am+hd}{c}\right) \times \left(\int_{\operatorname{Im} w=y} \frac{1}{(cw)^k}\, e\left(-\frac{m}{c^2w} - hw\right) dw\right) e(hz)$$

$$= e(mz) + \sum_{h\in\mathbb{Z}} p_k(m,h)\, e(hz)$$

where

$$p_k(m,h) = \sum_{c\geq 1} \sum_{\substack{0\leq d\leq c-1 \\ (c,d)=1}} e\left(\frac{am+hd}{c}\right) \int_{\operatorname{Im} w=y} \frac{1}{(cw)^k}\, e\left(-\frac{m}{c^2 w} - hw\right) dw$$

Note that $ad - bc = 1$ implies $ad \equiv 1 \mod c$ so $a \equiv \bar{d} \mod c$. So

$$\sum_{\substack{0 \le d \le c-1 \\ (c,d)=1}} e\left(\frac{am + hd}{c}\right) = \sum_{\substack{0 \le d \le c-1 \\ (c,d)=1}} e\left(\frac{\bar{d}m + hd}{c}\right)$$

$$= S(h, m; c)$$

$\big(= S(m, h; c)$ by changing d into $\bar{d}$ in the sum.$\big]$

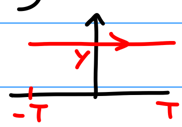
It remains to handle the complex integral.

Let

$$I_k(m, h; c) = \int_{\mathrm{Im}(w) = y} (cw)^{-k} e\left(-\frac{m}{c^2 w} - hw\right) dw.$$

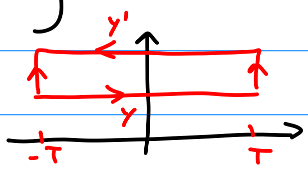
<u>Step 1</u>. The integral is independent of $y > 0$: write the integral as

$$\lim_{T \to \infty} \int \ldots$$



Then note that for $y' > 0$

$$\int \ldots = 0$$



where the contribution of the vertical segments from $\pm T + iy$ to $\pm T + iy'$ is

$$\le \int_y^{y'} \frac{1}{(c^2 |T + iu|)^k} e^{2\pi h u} \, du$$

$$\xrightarrow[T \to \infty]{} 0$$

So we get $\displaystyle\int_{\mathrm{Im}(w) = y'} (\ldots) = \int_{\mathrm{Im}(w) = y} (\ldots)$

Step 2. If $h \leq 0$ then $I_k(m,h;c) = 0$: at height y, the integral is bounded by

$$\int_{\mathbb{R}} \frac{1}{|(c^2(t+iy))|^k} \, dt \xrightarrow[y \to \infty]{} 0$$

[the exponential factors are ≤ 1 in modulus if $h \leq 0$].

Summarize again:

$$P_{m,k}(z) = e(mz) + \sum_{h\geq 1}\sum_{c\geq 1} S(h,m;c)\, I_k(m,h;c)\, e(hz)$$

First consequence: $P_{m,k}$ is holomorphic at ∞ [for $m \geq 0$] and even $P_{m,k} \in M_k^0$ if $m \geq 1$.

Step 3. Computation of $I_k(m,h;c)$

3.1. $\boxed{m = 0, \quad h \geq 1}$

Then

$$I_k(0,h;c) = \int_{\mathrm{Im}\, w = y} (cw)^{-k} e(-hw)\, dw \qquad \left(\begin{array}{l} v = hw \\ dv = h\, dw \end{array} \right.$$

$$= \frac{h^k}{c^k h} \int_{\mathrm{Im}\, v = y} v^{-k} e(-v)\, dv$$

$$= \frac{h^{k-1}}{c^k} \quad (\text{value depending on } k \text{ only})$$

Recall:

Definition (Gamma function). For $\mathrm{Re}(s) > 0$, $\Gamma(s) = \int_0^{+\infty} t^s e^{-t} \frac{dt}{t}$.

Lemma 3.

$$\int_{\mathrm{Im}\, v = y} v^{-k} e(-v)\, dv = \frac{(2\pi)^k}{i^k} \frac{1}{\Gamma(k)}.$$

(<u>Reference</u>: Whittaker and Watson, "A course of modern analysis", 1927; p° 245-246, Example 1, due to Laplace)

So
$$\boxed{I_k(0,h;c) = \frac{h^{k-1}}{c^k}\cdot\left(\frac{2\pi}{i}\right)^k \frac{1}{\Gamma(k)}.}$$

<u>3.2.</u> <u>$m \geq 1$</u>, <u>$h \geq 1$</u>

$$I_k(m,h;c) = \int_{\mathrm{Im}\, w = y} (cw)^{-k}\, e\left(-\frac{m}{c^2 w} - hw\right) dw$$

<u>Lemma 4.</u> For $m \geq 1$, $h \geq 1$, we have

$$I_k(m,h;c) = \frac{2\pi}{i^k c}\left(\frac{h}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

<u>Proof.</u> We can deduce this from Lemma 3 by expanding $e\left(-\frac{m}{c^2 w}\right)$ in power series:

$$e\left(\frac{-m}{c^2 w}\right) = \sum_{\ell \geq 0} \frac{1}{\ell!}\left(-\frac{m}{c^2 w}\right)^{\ell} (2i\pi)^{\ell}$$

So

$$I_k(m,h;c) = \sum_{\ell \geq 0} \frac{1}{\ell!} (-1)^{\ell} (2i\pi m)^{\ell} \frac{1}{c^{2(k+\ell)}} \int_{\mathrm{Im}\, w = y} \frac{1}{w^{k+\ell}}\, e(-hw)\, dw$$

$$\overset{\text{Lemma 3}}{=} \sum_{\ell \geq 0} \frac{(-1)^{\ell}}{\ell!} \frac{(2i\pi m)^{\ell}}{c^{2(k+\ell)}} h^{k-1+\ell} \left(\frac{2\pi}{i}\right)^{k+\ell} \frac{1}{\Gamma(\underbrace{k+\ell}_{k-1+1+\ell})}$$

$$= \sum_{\ell \geq 0} \frac{(-1)^{\ell}}{\ell!\,\Gamma(k-1+1+\ell)} \left(\frac{4\pi\sqrt{mn}}{2c}\right)^{k-1+2\ell} \times \left(\frac{h}{m}\right)^{\frac{k-1}{2}} \times \frac{2\pi}{c} \times \frac{1}{i^k}$$

$$= \frac{2\pi}{c\, i^k}\left(\frac{h}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

by the definition $J_\nu(z) = \sum_{\ell \geq 0} \frac{(-1)^{\ell}}{\ell!\,\Gamma(\nu+1+\ell)}\left(\frac{z}{2}\right)^{\nu+2\ell}$ of the Bessel function. $\square$

Note. The Bessel functions are characterized as solutions of certain ODE's of order 2 with specific initial conditions. This provides a way to prove many identities by checking that the "other side" also solves the same ODE.

Conclusion:

$m = 0$:

$$E_k(z) = 1 + \frac{(2\pi)^k}{i^k \underbrace{\Gamma(k)}_{=(k-1)!}} \sum_{n \geq 1} \left(n^{k-1} \sum_{c \geq 1} \frac{1}{c^k} S(0,n;c) \right) e(nz).$$

$m \geq 1$:

$$P_{m,k}(z) = \sum_{n \geq 1} p_k(m,n)\, e(nz) \qquad \text{where}$$

$$p_k(m,n) = \delta(m,n) + \frac{2\pi}{i^k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{c \geq 1} \frac{1}{c} S(m,n;c)\, J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

(Lemma 4).

What remains to conclude is:

Lemma 5. For $c \geq 1$, $n \geq 1$ we have

$$\underbrace{S(0,n;c)}_{\text{(Ramanujan sum)}} = \sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{nd}{c}\right) = \sum_{\delta \mid (n,c)} \delta\, \mu\left(\frac{c}{\delta}\right).$$

Möbius function

Ex. $c = p$ prime $\Longrightarrow$ $S(0,n;p) = \sum_{\substack{d \bmod p \\ d \neq 0}} e\left(\frac{nd}{p}\right)$

$$= \begin{cases} p-1 & \text{if } p \mid n \\ \underbrace{-1}_{=\mu(p)} & \text{if } p \nmid n \end{cases}$$

Proof. This can be done by Möbius inversion:

$$\sum_{d \bmod c} e\left(\frac{nd}{c}\right) = \sum_{\delta | c} \sum_{\substack{d \bmod c \\ (d,c)=\delta}} e\left(\frac{nd}{c}\right)$$

$$= \sum_{\delta | c} S(0, n; c/\delta)$$

but also $\displaystyle\sum_{d \bmod c} e\left(\frac{nd}{c}\right) = \begin{cases} c & \text{if } c | n, \\ 0 & \text{otherwise} \end{cases}$

and the lemma follows by Möbius inversion of the resulting identity

$$\sum_{\delta | c} S(0, n; \tfrac{c}{\delta}) = c\, \mathbb{1}_{c|n}$$

$\square$

Using this, we get

$$n^{k-1} \sum_{c \geq 1} \frac{1}{c^k} S(0, n; c)$$

$$= n^{k-1} \sum_{c \geq 1} \frac{1}{c^k} \sum_{\delta | (n,c)} \delta\, \mu\left(\frac{c}{\delta}\right)$$

$$= n^{k-1} \sum_{\delta | n} \delta \sum_{\substack{c \geq 1 \\ \delta | c}} \frac{1}{c^k} \mu\left(\frac{c}{\delta}\right)$$

$$= n^{k-1} \sum_{\delta | n} \delta \sum_{\alpha \geq 1} \frac{1}{(\delta \alpha)^k} \mu(\alpha)$$

$$= n^{k-1} \sum_{\delta | n} \frac{1}{\delta^{k-1}} \cdot \sum_{\alpha \geq 1} \frac{\mu(\alpha)}{\alpha^k}$$

$$= \frac{1}{\zeta(k)} \sum_{\delta | n} \left(\frac{n}{\delta}\right)^{k-1} = \frac{\sigma_{k-1}(n)}{\zeta(k)}.$$

Combined with what we had before, we get

$$E_k(z) = 1 + \frac{(2i\pi)^k}{\zeta(k)\, \Gamma(k)} \sum_{n \geq 1} \sigma_{k-1}(n)\, e(nz).$$

(for $z \in \mathbb{H}$)