D-MATH
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Distribution of primes

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Exercise Sheet 1

1. Show that
$$\int_2^x \frac{\mathrm{d}t}{\ln t} - \sum_{2\leq n\leq x} \frac{1}{\ln n} = O(\ln x).$$

2. Show that the Cramér model predicts the *Goldbach conjecture*, that is, that for every even integer $n \geq 6$, there exist primes p and q with $p+q=n$.

3. Fix an integer $k \geq 1$ and a prime p. For any distinct integers $h_1,\dots,h_k$, let $\nu_p(\{h_1,\dots,h_k\})$ denote the number of distinct residue classes modulo p occupied by the h_i's. Show that as $h \to \infty$,
$$\sum_{1\leq h_1<h_2<\cdots<h_k\leq h} \frac{1-\nu_p(\mathcal{H})/p}{(1-1/p)^k} \sim \sum_{1\leq h_1<h_2<\cdots<h_k\leq h} 1.$$

4. Show that the Cramér model predicts that for a fixed real number $\lambda > 0$ and non-negative integer k,
$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x : \pi(n+\lambda\log n) - \pi(n) = k\} = \frac{\lambda^k}{k!}e^{-\lambda}.$$