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Exercise Sheet 2

1. Recall that
$$\sum_{p\le x}\frac{\log p}{p}=\log x+O(1).$$

   a) Show that for some constant c,
$$\sum_{p\le x}\frac{1}{p}=\log\log x+c+O(1/\log x).$$

   b) Show that for some constant d,
$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=e^d\log x+O(1).$$
   How do the constants c and d relate to each other?

   c) Show that
$$\sum_{n\le x}\frac{\mu^2(n)}{\varphi(n)}\ge\sum_{k\le x}\frac{1}{k}\ge\log x.$$

2. Use Brun's pure sieve to prove an upper bound on the number of primes less than x. What can Brun's sieve say about lower bounds?

3. Prove the form of Möbius inversion that we used, namely that for any functions $f,g:\mathbb{N}\to\mathbb{C}$ with finite support,
$$g(n)=\sum_{n|d}f(d)$$
   if and only if
$$f(n)=\sum_{n|d}g(d)\mu(n/d).$$

4. Use Selberg's sieve to bound, for fixed even $N\ge 4$, the number of pairs of primes p_1,p_2 such that $p_1+p_2=N$. (See how far you can get on turning the bound into a nice shape).