Exercise Sheet 3

1. Use Selberg's sieve to show that for any positive integer m,

$$\#\{p \le x : p, p+2m \text{ both prime}\} \ll \frac{2m}{\varphi(2m)} \frac{x}{\log^2 x}.$$

2. The prime number theorem in arithmetic progressions states that for fixed q and for fixed a mod q with $(a,q)=1$,

$$\psi(x;a,q) := \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) = \frac{x}{\varphi(q)}(1 + O((\log x)^{-2})),$$

say. Consider the sieve problem for sums of two squares, where for $p \equiv 3 \mod 4$,

$$\mathcal{A}_p = \{n \le x : p^k \| n \text{ for } k \text{ odd}\}.$$

Using the prime number theorem in arithmetic progressions, show that the dimension of this sieve problem is $\frac{1}{2}$, i.e. that

$$\sum_{\substack{p \le z \\ p \equiv 3 \bmod 4}} \frac{\omega(p) \log p}{p} = \frac{1}{2} \log z + O(1).$$

3. The goal of this exercise is to prove *Romanoff's theorem*, which states that a positive proportion of positive integers n can be expressed as $n = p + 2^k$ where p is prime and k is a non-negative integer. (Note: this is a long exercise! Doing every part is of course optional.)

   (a) Let

   $$r(n) := \#\{(p,k) : k \ge 1, n = p + 2^k\},$$

   and let $B(x)$ denote the number of $n \le x$ such that such a decomposition exists, i.e. the number of $n \le x$ such that $r(n) \ge 1$. Show that

   $$\sum_{n \le x} r(n) \le \Big(B(x) \sum_{n \le x} r(n)^2\Big)^{1/2}.$$

   (Hint: Use Cauchy–Schwarz).

(b) Show that

$$\sum_{n\le x} r(n) \gg x$$

and that

$$\sum_{n\le x} r(n)^2 \le A := \#\{(p_1,p_2,k_1,k_2) : p_i \le x, 2^{k_i} \le x, p_1 + 2^{k_1} = p_2 + 2^{k_2}\}.$$

Conclude that $B(x) \gg \frac{x^2}{A}$.

(c) Apply the first problem with $2m = 2^{k_1} - 2^{k_2}$ to show that

$$A \ll x + \frac{x}{\log^2 x} \sum_{0<k_2<k_1\le \frac{\log x}{\log 2}} \frac{2^{k_1}-2^{k_2}}{\varphi(2^{k_1}-2^{k_2})}.$$

(d) Defining $\ell = k_1 - k_2$ and noting that there are $O(\log x)$ pairs k_1, k_2 with $k_1 - k_2 = \ell$, show that

$$\begin{aligned}\sum_{0<k_2<k_1\le \frac{\log x}{\log 2}} \frac{2^{k_1}-2^{k_2}}{\varphi(2^{k_1}-2^{k_2})} &\ll \log x \sum_{1\le \ell\le \frac{\log x}{\log 2}} \frac{2^\ell - 1}{\varphi(2^\ell-1)} \\ &= \log x \sum_{1\le \ell\le \frac{\log x}{\log 2}} \prod_{p\mid 2^\ell-1} \frac{p}{p-1} \\ &\ll \log x \sum_{1\le \ell\le \frac{\log x}{\log 2}} \sum_{d\mid 2^\ell-1} \frac{1}{d}.\end{aligned}$$

(e) Let $s(d)$ denote the order of d modulo 2. Show that

$$\sum_{1\le \ell\le \frac{\log x}{\log 2}} \sum_{d\mid 2^\ell-1} \frac{1}{d} \ll \log x \sum_{\substack{d\le x \\ d \text{ odd}}} \frac{1}{ds(d)}.$$

Conclude that

$$A \ll x + x \sum_{\substack{d\le x \\ d \text{ odd}}} \frac{1}{ds(d)}.$$

(f) The next few parts will show that the sum over d converges when extended to all odd positive integers. Show that $s(d) \gg \log d$, and use this to show that for all $k \ge 1$

$$t_k := \sum_{\substack{d\in\mathbb{N} \\ d \text{ odd} \\ s(d)\le k}} \frac{1}{d}$$

is finite. Conclude that

$$\sum_{\substack{d\in\mathbb{N} \\ d \text{ odd}}} \frac{1}{ds(d)} = \sum_{k\ge 1} \frac{t_k - t_{k-1}}{k} = \sum_{k\ge 1} \frac{t_k}{k(k+1)}.$$

(g) Define $N_k := \prod_{i \le k}(2^i - 1)$. Show that if $s(d) \le k$, then $d | N^k$, and that $N_k < 2^{k^2}$. Conclude that

$$t_k \le \sum_{d | N_k} \frac{1}{d} \ll \log k.$$

(h) Combining the results of the previous parts, show that

$$\sum_{\substack{d \in \mathbb{N} \\ d \text{ odd}}} \frac{1}{d s(d)}$$

converges to a constant. Conclude that $A \ll x$ and that $B(x) \gg x$, as desired.