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Exercise Sheet 3

1. Recall that $\tau_k(n)$ denotes the k-fold divisor function of n, that is, it is defined by

$$\tau_k(n) := \#\{n_1, \dots, n_k \in \mathbb{N}_{\geq 1} : n_1 \cdots n_k = n\}.$$

   (a) Show that for any $k \geq 2$, as $D \to \infty$

$$\sum_{n<D} \tau_k(n) \ll D(\log D)^{k-1}.$$

   (b) Let $k \geq 2$. Show that

$$\sum_{q<x^{1/2}e^{-\sqrt{\log x}}} \mu^2(q)\tau_{3k}^2(q)\frac{x}{\varphi(q)} \ll x(\log x)^{(9k^2+1)}.$$

2. Recall that if there exists an $F \in \mathcal{F}_k$ with $\frac{kJ(F)}{I(F)} > 4(m-1)$, then any admissible k-tuple contains m primes infinitely often. This proves that $\liminf p_{n+m-1} - p_n < \infty$ for all $m \geq 2$ as long as we know that $M_k := \sup_{F \in \mathcal{F}_k} \frac{kJ(F)}{I(F)} \to \infty$ as $k \to \infty$.

   Write down an explicit function $F \in \mathcal{F}_{1500}$ such that $\frac{kJ(F)}{I(F)} > 4$, and prove that $\frac{kJ(F)}{I(F)} > 4$ for the F you find. (Hint: Use the form that we discussed in lecture).

3. (a) For $k \geq 2$, show that $p_{\pi(k)+1}, \dots, p_{\pi(k)+k}$ forms an admissible k-tuple.

   (b) The prime number theorem states that

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

   Show that the prime number theorem implies that

$$p_k = k\log k + k\log\log k - k + o(k).$$

   (Hint: use partial summation.)

   (c) Let $H(k)$ be the minimum of $h_k - h_1$ taken over all admissible k-tuples $\{h_1, \dots, h_k\}$ with $h_1 < \cdots < h_k$. Show that

$$H(k) \leq k\log k + k\log\log k - k + o(k).$$

   (d) State (and prove) an explicit upper bound for $H(1500)$. Deduce an explicit upper bound for prime gaps.