Exercise Sheet 5

1. Recall that $S(N,\alpha)$ is the sum defined as
$$S(N,\alpha) := \sum_{n\le N} e(n\alpha)\Lambda(n).$$
For a constant $c>0$ and an integer N, let $T_N(c)\subset[0,1]$ be defined as
$$T_N(c) = \{\alpha\in[0,1] : |S(N,\alpha)|\ge cN\}.$$
Show that $|T_N(c)| \le \frac{\log N}{c^2 N}(1+o(1))$.

2. Let $\alpha\in\mathbb{R}$.
   (a) Show that $\left|\sum_{n\le N} e(\alpha n)\right| \le N$ and that
$$\sum_{n\le N} e(\alpha n) = \frac{1-e(\alpha(N+1))}{1-e(\alpha)}.$$
   (b) Show that for all $\alpha\in(0,1/2]$,
$$\left|\frac{1}{1-e^{2\pi i\alpha}}\right| < \frac{1}{2\alpha}.$$
   (Hint: square both sides, and then expand the left-hand side into a real function of α. Then use calculus.)
   (c) Conclude that
$$\Big|\sum_{n\le N} e(\alpha n)\Big| \le \min\{N, \|\alpha\|^{-1}\}.$$
   (d) Use partial summation to show that
$$\Big|\sum_{n\le N}\frac{e(\alpha n)}{\log n}\Big| \ll \frac{\min\{N,\|\alpha\|^{-1}\}}{\log N} + \frac{\|\alpha\|^{-1}}{\log(\|\alpha\|^{-1})}.$$

3. Let α be an irrational number. Show that for every $\beta\in[0,1)$ and every $\varepsilon>0$, there is a positive integer n such that $|\beta - \|n\alpha\|| < \varepsilon$.