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Exercise Sheet 6

1. In class, we proved Vinogradov's theorem, i.e. that for N odd, for all $A > 0$,

$$\sum_{\substack{n_1,n_2,n_3 \le N \\ n_1+n_2+n_3=N}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) = \frac{\mathfrak{S}_N}{2} N^2 + O_A\left(\frac{N^2}{(\log N)^A}\right),$$

   where

$$\mathfrak{S}_N = \prod_{p \mid N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p \nmid N}\left(1 + \frac{1}{(p-1)^3}\right).$$

   Using Vinogradov's theorem, show that there exists some M such that for all odd $N \ge M$, N can be written as a sum of three primes.

2. This (long) exercise will be dedicated to showing that there are infinitely many *ternary arithmetic progressions* in the primes, i.e. there are infinitely many triples n_1, n_2, n_3 such that $n_3 - n_2 = n_2 - n_1$ and each of the n_i is prime.

   (a) Define $T(x)$ to be the following weighted count of ternary arithmetic progressions, all of whose entries are less than x:

$$T(x) := \sum_{\substack{n_1,n_2,n_3 \le x \\ n_3-n_2=n_2-n_1}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3).$$

   Show that

$$T(x) = \int_0^1 S(x,\alpha)^2 S(x,-2\alpha)\,\mathrm{d}\alpha,$$

   where $S(x,\alpha) = \sum_{n \le x} e(n\alpha)\Lambda(n)$.

   (b) We will use the same division into major and minor arcs as for Vinogradov's theorem (you should convince yourself intuitively that this is a reasonable choice). That is, the major arcs $\mathfrak{M}$ will be given by

$$\mathfrak{M} = \bigcup_{1 \le q \le \mathfrak{L}} \bigcup_{\substack{1 \le a \le q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{\mathcal{L}}{x}, \frac{a}{q} + \frac{\mathcal{L}}{x}\right],$$

   where $\mathcal{L} = (\log x)^{2A+10}$, and the minor arcs will be $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$. Using Vinogradov's bound (Proposition 8.13 in the notes), show that for any $\alpha \in \mathfrak{m}$, $|S(x,-2\alpha)| \ll x/(\log x)^{A+1}$.

(c) Show that

$$\int_{\mathfrak{m}} S(x,\alpha)^2 S(x,-2\alpha)\mathrm{d}\alpha \ll \frac{x^2}{(\log x)^A}.$$

(d) Recall that for $\alpha = \beta + a/q$, for $q \leq \mathcal{L}$, and $|\beta| \leq \mathcal{L}/x$,

$$S(x,\alpha) = \frac{\mu(q)}{\varphi(q)} \int_2^x e(\beta t)\mathrm{d}t + O(qxe^{-c\sqrt{\log x}})$$

for some constant c. Derive an expression for $S(x,-2\alpha)$ (hint: the answer will be different if q is even or odd).

(e) Show that

$$\int_{\mathfrak{M}} S(x,\alpha)^2 S(x,-2\alpha)\mathrm{d}\alpha = \sum_{\substack{1\leq q\leq \mathcal{L}\\ 1\leq a\leq q\\ (a,q)=1}} (-1)^{q+1}\frac{\mu(q)}{\varphi(q)^3}\int_{-\mathcal{L}/x}^{\mathcal{L}/x}\left(\int_2^x e(\beta t)\mathrm{d}t\right)^2\int_2^x e(-2\beta t)\mathrm{d}t\mathrm{d}\beta$$
$$+ O\left(\mathcal{L}^3 N^2 e^{-c\sqrt{\log N}}\right).$$

(f) Show that

$$\sum_{\substack{1\leq q\leq \mathcal{L}\\ 1\leq a\leq q\\ (a,q)=1}} (-1)^{q+1}\frac{\mu(q)}{\varphi(q)^3} = 2\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^2}\right) + O\left(\frac{\log \mathcal{L}}{\mathcal{L}}\right).$$

(g) Show that

$$\int_{-\mathcal{L}/x}^{\mathcal{L}/x}\left(\int_2^x e(\beta t)\mathrm{d}t\right)^2\int_2^x e(-2\beta t)\mathrm{d}t\mathrm{d}\beta = \frac{x^2}{2} + O(x^2/\mathcal{L}^2).$$

(h) Conclude that

$$T(x) = x^2\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^2}\right) + O_A(x^2/(\log x)^A),$$

and explain why this implies that there are infinitely many ternary arithmetic progressions in the primes.