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Exercise Sheet 7

1. (see Koukoulopoulos, exercise 23.1) The purpose of this exercise is to prove the following: for $x \geq 2$ and for $\alpha \in \mathbb{R}$ and a reduced fraction a/q such that $|\alpha - a/q| \leq 1/q^2$,
$$\sum_{n \leq x} d(n) e(\alpha n) \ll (\sqrt{x} + q + x/q) \log x,$$
where $d(n)$ denotes the number of divisors of n.

   a) Show that
$$\sum_{n \leq x} d(n) e(\alpha n) = \sum_{k \leq \sqrt{x}} \sum_{\ell \leq x/k} e(\alpha k \ell) + \sum_{\ell \leq \sqrt{x}} \sum_{k \leq x/\ell} e(\alpha k \ell) - \sum_{\ell, k \leq \sqrt{x}} e(\alpha k \ell).$$

   b) Show (for example using Lemma 9.12 from the lecture notes) that
$$\sum_{\ell, k \leq \sqrt{x}} \ll (\sqrt{x} + q + x/q) \log x.$$

   c) Show that
$$\sum_{k \leq \sqrt{x}} \sum_{\ell \leq x/k} e(\alpha k \ell) \ll (\sqrt{x} + q + x/q) \log x.$$
   (Hint: can this be done using our type I bound or our type II bound?)

   d) Deduce the desired bound on $\sum_{n \leq x} d(n) e(\alpha n)$.

2. (Heath-Brown's identity; see Koukoulopoulos, exercise 23.5). Let $k \in \mathbb{N}$, $x \geq 1$ and $V \geq x^{1/k}$. For $n \leq x$, show that
$$\Lambda(n) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (\log * \underbrace{1 * \cdots * 1}_{j-1 \text{ times}} * \underbrace{\mu_{\leq V} * \cdots * \mu_{\leq V}}_{j \text{ times}})(n).$$
(Hint: Let $f = \mu_{\leq V} * 1$ and let $g = \mu_{>V} * 1$. On the one hand, we have $\Lambda * \underbrace{g * \cdots * g}_{k \text{ times}}(n) = 0$ if $n \leq x$. On the other hand, $g = 1_{n=1} - f$.)