# Practice Exam

1. Use partial summation to show that, as $x \to \infty$,

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O(1/x),$$

where

$$\gamma := 1 - \int_1^\infty \frac{\{t\}}{t^2} \mathrm{d}t$$

and $\{t\} := t - \lfloor t \rfloor$ for $t \in \mathbb{R}$.

2. Consider the polynomials $f_1(n)$ and $f_2(n)$ with

$$f_1(n) = n^2 + n + 1, \qquad f_2(n) = 3n^2 - 5n + 1.$$

a) Show that $f_1$ and $f_2$ have no roots mod 2, and compute the roots of $f_1$ and $f_2$ mod 3.

b) Do you expect there to be infinitely many $n \in \mathbb{N}$ such that $f_1(n)$ and $f_2(n)$ are both prime? Why or why not?

3. Define $S(x, \alpha) := \sum_{n \le x} \tau(n) e(n\alpha)$, where $\tau(n)$ denotes the divisor function and $e(\vartheta) := e^{2\pi i \vartheta}$. Compute $S(x, 1/2)$.

4. The goal of this exercise is, given a large even integer $N$, to provide an upper bound on the number of representations of $N$ as a sum of two primes, using Selberg's sieve. Recall the general statement of Selberg's sieve that we saw in lecture:

**Theorem 1** (Selberg's sieve). *Let $S(\mathcal{A}, z)$ be a sieve problem counting elements of a sequence $\mathcal{A} = (a_n)$ with $n \le x$, all of which are not divisible by any prime $p \le z$. Let $P(z)$ denote the product of all primes $p \le z$. Assume that $g(d)$ is a multiplicative function and $r_d$ is a remainder such that $\#\mathcal{A}_d := \#\{n \le x : d | a_n\} = xg(d) + r_d$. Then we have*

$$S(\mathcal{A}, z) \le \frac{x}{S(z)} + R(z),$$

*where*

$$S(z) = \sum_{d \le z} \frac{\mu^2(d)}{(\mu * (1/g))(d)}$$

*and*

$$R(z) = \sum_{\substack{d_1, d_2 \le z \\ d_1, d_2 | P(z)}} |r_{[d_1, d_2]}|.$$

a) Fix $N$ even. Define a sieve problem $S(\mathcal{A}, z)$, depending on $N$, that counts the number of $n \leq N$ such that $n$ and $N - n$ are both relatively prime to $z$. (That is, write down the sequence $\mathcal{A} = (a_n)$, and choose $x = N$). Find a multiplicative function $g(d)$ such that

$$\#\mathcal{A}_d = Ng(d) + r_d,$$

where $r_d$ satisfies

$$|r_d| \ll \tau(d).$$

b) Find the value of $\varkappa \geq 0$ such that

$$\sum_{p \leq z} g(p) \log p = \varkappa \log z + O(1).$$

This implies (which you may assume without proof) that $S(z)^{-1} \ll (\log z)^{\varkappa}$.

c) Show that

$$R(z) \ll z^2 \log^2 z.$$

d) Given the bounds on $S(z)$ and $R(z)$ from the previous parts, provide an (asymptotic) upper bound on $S(\mathcal{A}, z)$. What choice of $z$ optimizes this bound?