Prof. Dr. Alessandra Iozzi
Konstantin Andritsch

Exercise Sheet 1

Exercise 1 (Compact-Open Topology). Let X, Y, Z be a topological space, and denote by $C(Y, X) \coloneqq \{f : Y \to X \text{ continuous}\}$ the set of continuous maps from Y to X. The set $C(Y, X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K, U) \coloneqq \{f \in C(Y, X) \mid f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Prove the following useful facts about the compact-open topology.

If Y is locally compact[1] , then:

a) The evaluation map $e : C(Y, X) \times Y \to X, e(f, y) \coloneqq f(y)$, is continuous.

b) A map $f : Y \times Z \to X$ is continuous if and only if the map

$$\hat{f} : Z \to C(Y, X), \quad \hat{f}(z)(y) = f(y, z),$$

is continuous.

Exercise 2 (General Linear Group $\mathrm{GL}(n, \mathbb{R})$). The general linear group

$$\mathrm{GL}(n, \mathbb{R}) \coloneqq \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\} \subseteq \mathbb{R}^{n\times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$\begin{aligned} j :\ \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Homeo}(\mathbb{R}^n), \\ A &\mapsto (x \mapsto Ax). \end{aligned}$$

a) Show that $j(\mathrm{GL}(n, \mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset C(\mathbb{R}^n, \mathbb{R}^n)$ is endowed with the compact-open topology.

b) If we identify $\mathrm{GL}(n, \mathbb{R})$ with its image $j(\mathrm{GL}(n, \mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with the usual topology coming from the inclusion $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$.
Hint: Exercise 1 can be useful here.

Exercise 3 ($\mathrm{O}(p, q)$). We consider the orthogonal group $\mathrm{O}(p, q)$ of signature $p, q \geq 1$.

a) Show that the connected component of the group $O(1, 1)$ containing the identity is homeomorphic to $\mathbb{R}$.

b) Show that for all $p, q \geq 1$, $\mathrm{O}(p, q)$ has a subgroup isomorphic to $\mathbb{R}$.

[1] A subset $C \subseteq Y$ that contains an open subset $U \subseteq Y$ with $y \in U \subseteq C \subseteq Y$ is called a *neighborhood of* $y \in Y$. Then Y is called *locally compact* if for every $y \in Y$ there is a set $\mathcal{D}$ of compact neighborhoods of y such that every neighborhood of y contains an element of $\mathcal{D}$ as a subset.

Exercise 4 (Isometry Group Iso(X)). Let (X,d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x,y) \quad \text{for all } x, y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem, see Appendix A.2 in Prof. Alessandra Iozzi's book.

Exercise† 5 (Homeomorphism Group Homeo(X)).

a) Let X be a *compact* Hausdorff space. Show that $(\mathrm{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.

b) The objective of this exercise is to show that $(\mathrm{Homeo}(X), \circ)$ will not necessarily be a topological group if X is only locally compact.

Consider the "middle thirds" Cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1]$$

in the unit interval. We define the sets $U_n = C \cap [0, 3^{-n}]$ and $V_n = C \cap [1-3^{-n}, 1]$. Further we construct a sequence of homeomorphisms $h_n \in \mathrm{Homeo}(C)$ as follows:

- $h_n(x) = x$ for all $x \in C \setminus (U_n \cup V_n)$,
- $h_n(0) = 0$,
- $h_n(U_{n+1}) = U_n$,
- $h_n(U_n \setminus U_{n+1}) = V_{n+1}$,
- $h_n(V_n) = V_n \setminus V_{n+1}$.

These restrict to homeomorphisms $h_n|_X$ on $X := C \setminus \{0\}$.

Show that the sequence $(h_n|_X)_{n\in\mathbb{N}} \subset \mathrm{Homeo}(X)$ converges to the identity on X but the sequence $((h_n|_X)^{-1})_{n\in\mathbb{N}} \subset \mathrm{Homeo}(X)$ of their inverses does not!

Remark: However, if X is locally compact and *locally connected* then $\mathrm{Homeo}(X)$ is a topological group.

c) Let $\mathbb{S}^1 \subset \mathbb{C} \setminus \{0\}$ denote the circle. Show that $\mathrm{Homeo}(\mathbb{S}^1)$ is not locally compact.

Remark: In fact, $\mathrm{Homeo}(M)$ is not locally compact for any manifold M.

Exercise 6. Locally Compact Hilbert Spaces are Finite-Dimensional Let $\mathcal{H}$ be a Hilbert space a field $k = \mathbb{R}$ or $\mathbb{C}$. Show that $\mathcal{H}$ is locally compact if and only if it is finite-dimensional.

Exercise 7 (Unitary Operators). Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H})$ its group of unitary operators. Show that the weak operator topology coincides with the strong operator topology on $U(\mathcal{H})$.

Hint: Recall that a sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *weak operator topology* if

$$\lambda(T_n x) \to \lambda(Tx) \quad (n \to \infty)$$

for every linear functional $\lambda \in \mathcal{H}^*$ and every $x \in \mathcal{H}$.

A sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *strong operator topology* if

$$T_n x \to Tx \quad (n \to \infty)$$

for every $x \in \mathcal{H}$.

Exercise 8 (p-adic Integers $\mathbb{Z}_p$)**.** Let $p \in \mathbb{N}$ be a prime number. Recall that the p-adic integers $\mathbb{Z}_p$ can be seen as the subspace

$$\left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} : a_{n+1} \equiv a_n \ (\mathrm{mod}\ p^n) \right\}$$

of the infinite product $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}_p$ carrying the induced topology. Note that each $\mathbb{Z}/p^n\mathbb{Z}$ carries the discrete topology and $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is endowed with the resulting product topology.

a) Show that the image of $\mathbb{Z}$ via the embedding

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathbb{Z}_p, \\ x &\mapsto (x\ (\mathrm{mod}\ p^n))_{n\in\mathbb{N}} \end{aligned}$$

is dense. In particular, $\mathbb{Z}_p$ is a compactification of $\mathbb{Z}$.

b) Show that the 2-adic integers $\mathbb{Z}_2$ are homeomorphic to the “middle thirds” cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0, 2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0, 1].$$