Exercise Sheet 2

Exercise 1 (Transitive Group Actions)**.** Let G be a topological group, X a topological space and $\mu : G\times X \to X$ a continuous transitive group action, i.e. for any two $x, y \in X$ there is $g \in G$ such that $\mu(g, x) = g \cdot x = y$.

a) Show that if G is compact then X is compact.

b) Show that if G is connected then X is connected.

Exercise 2 (Examples of Haar Measures)**.**

a) Let us consider the *three-dimensional Heisenberg group* $H = \mathbb{R} \rtimes_\eta \mathbb{R}^2$, where $\eta : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is defined by

$$\eta(x)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z + xy \end{pmatrix},$$

for all $x, y, z \in \mathbb{R}$. Thus the group operation is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and it is easy to see that it can be identified with the matrix group

$$H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Verify that the Lebesgue measure is the Haar measure of $\mathbb{R} \rtimes_\eta \mathbb{R}^2$ and that the group is unimodular.

b) Let

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.$$

Show that $\frac{da}{a^2}\, db$ is the left Haar measure and $da\, db$ is the right Haar measure. In particular, P is *not* unimodular.

c) Let $G := \mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ denote the group of invertible matrices over $\mathbb{R}$. Let λ_{n^2} denote the Lebesgue measure on $\mathbb{R}^{n^2}$. Prove that

$$\mathrm{d}m(x) := |\mathrm{det} x|^{-n}\, \mathrm{d}\lambda_{n^2}(x)$$

defines a bi-invariant (i.e. left- and right-invariant) Haar measure on G.

d) Let $G = \mathrm{SL}_n(\mathbb{R})$ denote the group of matrices of determinant 1 in $\mathbb{R}^{n\times n}$. For a Borel subset $B \subseteq \mathrm{SL}_n(\mathbb{R})$ define

$$m(B) := \lambda_{n^2}\big(\{tg; g \in B, t \in [0, 1]\}\big).$$

Show that m is a well-defined bi-invariant Haar measure on $\mathrm{SL}_n(\mathbb{R})$.

e) Let G denote the $ax + b$ group defined as

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} ; a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$

Note that every element in G can be written in a unique fashion as a product of the form:

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$, which yields a coordinate system $\mathbb{R}^\times \times \mathbb{R} \leftrightarrow G$. Prove that

$$\mathrm{d}m(\alpha, \beta) = \frac{1}{|\alpha|}\, \mathrm{d}\alpha\, \mathrm{d}\beta$$

defines a left Haar measure on G. Calculate $\Delta_G(\alpha, \beta)$ for $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$.

Exercise 3. Haar measures on profinite groups Let $(I, \leq)$ be a directed (index) set and let $\{f_i^j \colon G_j \to G_i \,|\, i \leq j\}$ be a projective system of discrete finite groups $\{G_i \,|\, i \in I\}$, i.e. $f_i^j \colon G_j \to G_i, i \leq j$, are group homomorphisms such that $f_i^j \circ f_j^k = f_i^k$ for all $i \leq j \leq k$. We define its *inverse limit* as the subgroup

$$\varprojlim G_i \coloneqq \left\{ (g_i)_{i\in I} \in \prod_{i\in I} G_i \,\middle|\, f_i^j(g_j) = g_i \quad \forall i \leq j \right\} \subseteq \prod_{i\in I} G_i.$$

The inverse limit $\varprojlim G_i$ is a closed subgroup of the compact group $\prod_{i\in I} G_i$ and thus it is compact, too. Any topological group that is the inverse limit of a projective system of discrete finite groups is called a *profinite group*.

Let $\overline{G} = \varprojlim G_i$ be a profinite group and denote $\overline{\pi}_j \colon \overline{G} \to G_j, (g_i)_{i\in I} \mapsto g_j$ for every $j \in I$.

1) Show that the kernels $\{K_j \coloneqq \ker(\overline{\pi}_j)\}_{j\in I}$ form a neighborhood basis of the identity consisting of compact open sets.

Remark: This shows that every profinite group is locally compact.

2) Let μ be a Haar measure on $\overline{G}$ normalized such that $\mu(\overline{G}) = 1$.

Show that
$$\mu(K_j) = \frac{1}{|\overline{G} \colon K_j|} = \frac{1}{|\mathrm{im}(\overline{\pi}_j)|}.$$

3) Let $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ be the p-adic integers, let μ be a normalized Haar measure, and set $C_m \coloneqq \{(a_i)_{i\in\mathbb{N}}\mathbb{Z}_p \,|\, 0 = a_1, \ldots, 0 = a_m\}$.

Compute $\mu(C_m)$ using part 2.

Exercise 4 $(\mathrm{Aut}(\mathbb{R}^n, +) \cong \mathrm{GL}(n, \mathbb{R}))$**.** For a topological group G, we denote by $\mathrm{Aut}(G)$ the group of bijective, continuous homomorphisms of G with continuous inverse. Consider the locally compact Hausdorff group $G = (\mathbb{R}^n, +)$ where $n \in \mathbb{N}_0$.

a) Show that $\mathrm{Aut}(G)$, i.e. the group of bijective homomorphisms which are homeomorphisms as well, is given by $\mathrm{GL}_n(\mathbb{R})$.

b) Show that $\mathrm{mod} : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is given by $\alpha \mapsto |\mathrm{det}\alpha|$.

c) Prove that there exists a discontinuous, bijective homomorphism from the additive group $(\mathbb{R}, +)$ to itself.