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Exercise Sheet 3

Exercise 1. For each of the following locally compact Hausdorff groups, give an example of a lattice or prove that it does not admit a lattice.

a) The free group on 2 generators with the discrete topology.

b) $G = \mathrm{SO}(n, \mathbb{R})$.

c) $G = (\mathbb{R}_{>0}, \cdot)$.

d) $G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$, see exercise 2b) on Sheet 2.

e) The Heisenberg group
$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$
see exercise 2a) on Sheet 2.

Exercise 2 (Regular Subgroups are closed)**.** Let G be a Lie group, $H \leq G$ a subgroup that is also a regular submanifold. Prove that H is a closed subgroup of G.

Exercise 3 (The Matrix Lie Groups $O(p,q)$ and $U(p,q)$)**.** Let $p, q \in \mathbb{N}$ and $n = p + q$.

a) We define the (indefinite) symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p, q) on $\mathbb{R}^n$ to be
$$\langle v, w \rangle_{p,q} := v_1 w_1 + \cdots + v_p w_p - v_{p+1} w_{p+1} - \cdots - v_{p+q} w_{p+q}$$
for all $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. As the orthogonal group $O(n)$ is defined to be the group of matrices that preserve the standard Euclidean inner product we may now define $O(p, q)$ to be the group of matrices that preserve the above bilinear form:
$$O(p, q) := \{ A \in \mathrm{GL}(n, \mathbb{R}) : \langle Av, Aw \rangle_{p,q} = \langle v, w \rangle_{p,q} \quad \forall v, w \in \mathbb{R}^n \}.$$
Show that $O(p, q)$ is a Lie group using the inverse function theorem/constant rank theorem. What is its dimension?

b) Similarly we may define the following symmetric sesquilinear form on $\mathbb{C}^n$
$$\langle w, z \rangle_{p,q} := \bar{w}_1 z_1 + \cdots + \bar{w}_p z_p - \bar{w}_{p+1} z_{p+1} - \cdots - \bar{w}_{p+q} z_{p+q}$$
for all $w = (w_1, \ldots, w_n), z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and
$$U(p, q) = \{ A \in \mathrm{GL}(n, \mathbb{C}) : \langle Aw, Az \rangle_{p,q} = \langle w, z \rangle_{p,q} \quad \forall w, z \in \mathbb{C}^n \}.$$
Show that $U(p, q)$ is a (real) Lie group using the inverse function theorem/constant rank theorem. What is its (real) dimension?

Exercise 4 (Differential of det)**.** We consider the determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function
$$D_I \det = \mathrm{tr}.$$

Exercise 5 (Tangent space of manifold)**.** Let M be a smooth n-dimensional manifold and $p \in M$. Show that if (U, φ) is any chart at p with $\varphi(p) = 0$, then the map
$$\mathbb{R}^n \to T_p M, \quad v \mapsto (f \mapsto D_0(f \circ \varphi^{-1})(v))$$
is a vector space isomorphism.