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Exercise Sheet 5

Exercise 1 (Abstract Subgroups as Lie Subgroups)**.** Let H be an abstract subgroup of a Lie group G and let $\mathfrak{h}$ be a subspace of the Lie algebra $\mathfrak{g}$ of G. Further let $U \subseteq \mathfrak{g}$ be an open neighborhood of $0 \in \mathfrak{g}$ and let $V \subseteq G$ be an open neighborhood of $e \in G$ such that the exponential map $\exp : U \to V$ is a diffeomorphism satisfying $\exp(U \cap \mathfrak{h}) = V \cap H$. Show that the following statements hold:

a) H is a Lie subgroup of G with the induced relative topology;

b) $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;

c) $\mathfrak{h}$ is the Lie algebra of H.

Remark: This is precisely the lemma we saw in class but did not prove.

Exercise 2 (Quotients of Lie groups)**.** Let G be a Lie group and let $K \leq G$ be a closed normal subgroup.

Show that G/K can be equipped with a Lie group structure such that the quotient map $\pi\colon G \to G/K$ is a surjective Lie group homomorphism with kernel K.

Exercise 3 ($\mathrm{Z}(G) = \mathrm{Ker}(\mathrm{Ad})$)**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Use the fundamental relation that

$$g \exp(tX) g^{-1} = \exp(t \operatorname{Ad}_g(X))$$

for all $g \in G, t \in \mathbb{R}$ and $X \in \mathfrak{g}$ to prove the following.

(1) If G is connected, then the center $\mathrm{Z}(G)$ of G equals the kernel of the adjoint representation.

(2) If G is connected, $\mathrm{Z}(G)$ is a closed subgroup and

$$\mathrm{Lie}(\mathrm{Z}(G)) = \mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \colon \forall Y \in \mathfrak{g}, [X, Y] = 0\}.$$

Exercise 4 (The adjoint representation ad)**.** Let V be a vector space over a field k.

a) Show that the vector space of endomorphisms

$$\mathfrak{gl}(V) := \{A\colon V \to V \text{ linear}\}$$

is a Lie algebra with the Lie bracket given by the commutator

$$[A, B] := AB - BA$$

for all $A, B \in \mathfrak{gl}(V)$.

b) Let $\mathfrak{g}$ be a Lie algebra over k. The *adjoint representation*

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is defined as $\mathrm{ad}(X)(Y) := [X, Y]$ for all $X, Y \in \mathfrak{g}$. Show that ad is a Lie algebra homomorphism.

Exercise 5. Adjoint of nilpotent elements Let $\mathfrak{g} \leq \mathfrak{gl}_n(\mathbb{C})$ be a Lie subalgebra.

Show that, if $X \in \mathfrak{g}$ is nilpotent then $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.