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Exercise Sheet 6

Exercise 1 (Characterization of Solvability via ad)**.** Let $\mathfrak{g}$ be a Lie algebra. Show that $\mathfrak{g}$ is solvable if $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}^{(2)})$ is solvable.

Exercise 2 (Cartan's criterion for solvability)**.** Let $\mathfrak{g}$ be a Lie algebra with Killing form $B_{\mathfrak{g}}$.

Show that $\mathfrak{g}$ is solvable if and only if $B_{\mathfrak{g}}|_{\mathfrak{g}^{(1)}\times\mathfrak{g}^{(1)}} = 0$.

Hint: One direction is an easy verification. For the other direction you can use a theorem from class in conjunction with the fact that a Lie algebra $\mathfrak{g}$ with an ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is solvable if and only if both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

Exercise 3 (Engel's theorem)**.** The objective of this exercise is to give a proof of Engel's theorem: Let $\mathfrak{g}$ be a Lie algebra over a field k, let V be a finite-dimensional vector space over k, and let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation such that $\pi(X) \in \mathfrak{gl}(V)$ is nilpotent for every $X \in \mathfrak{g}$. Then $\pi(\mathfrak{g})$ has a common null vector $v_0 \neq 0$ in V.

a) Argue why one may assume without loss of generality that the representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is injective. This will then enable us to identify $\mathfrak{g}$ with its image $\pi(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$.

b) Show that if $X \in \mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a nilpotent element then so is $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$.

c) Using part a) and b) prove Engel's theorem by induction on $\dim \mathfrak{g}$.

   Hint: The base case $\dim \mathfrak{g} = 1$ is elementary. For the inductive step prove the following two statements using the inductive hypothesis:

   i) One can write $\mathfrak{g} = \mathfrak{h} \oplus kX_0$ where $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is an ideal and $X_0 \in \mathfrak{g}$.

   ii) One can find a common null vector for X_0 in the space of common null vectors for $\mathfrak{h}$.

Exercise 4 (Examples of solvable and nilpotent groups)**.** Compute the derived series and the central series of the Lie groups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b \neq 0 \right\}, \qquad H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \right\} \qquad \text{and} \qquad \mathrm{SL}(2, \mathrm{Hom})$$

to decide whether they are are solvable and/or nilpotent. Find all weights for the inclusion-representations $\rho_G\colon G \to \mathrm{GL}(2,\mathbb{C})$, $\rho_H\colon H \to \mathrm{GL}(3,\mathbb{C})$ and $\rho_{\mathrm{SL}(2,\mathbb{R})}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(2,\mathbb{C})$.

Exercise 5. Direct sums of simple ideals Let $\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i$ be the direct sum of simple ideals. Then any ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{j\in J} \mathfrak{g}_j$ with $J \subset I$.

Remark: This implies immediately:

(i) Any semisimple Lie algebras has a finite number of ideals.

(ii) Any connected semisimple Lie group with finite center has a finite number of connected normal subgroups.