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Introduction to Lie Groups

FS 2025

Extra Exercise Sheet

Exercise 1 (Invariant bilinear forms)**.** Let $k = \mathbb{R}$, or $\mathbb{C}$. Let V be a finite-dimensional vector space over k, let $f : V \times V \to k$ be a bilinear form and let $G \leq \mathrm{GL}(V)$ be a Lie subgroup with Lie algebra $\mathfrak{g} \leq \mathfrak{gl}(V) = \mathrm{End}(V)$.

We say that $f : V \times V \to k$ is G-invariant if

$$f(g \cdot v, g \cdot w) = f(v, w)$$

for all $g \in G, v, w \in V$. We say that $f : V \times V \to k$ is $\mathfrak{g}$-invariant if

$$f(X \cdot v, w) = -f(v, X \cdot w)$$

for all $X \in \mathfrak{g} \leq \mathrm{End}(V), v, w \in V$.

1) Assuming that $G \leq \mathrm{GL}(V)$ is connected show that f is G-invariant if and only if f is $\mathfrak{g}$-invariant.

2) Let $\mathfrak{g}$ be a Lie algebra over k, let $\rho : \mathfrak{g} \to \mathfrak{gl}(n, k)$ be a representation and consider the corresponding trace form $B_\rho(X, Y) = \mathrm{tr}(\rho(X)\rho(Y))$ of ρ. Show that B_ρ is $\mathrm{ad}(\mathfrak{g})$-invariant.

Exercise 2 (Killing form of $\mathfrak{sl}(2, \mathbb{R})$)**.** Choose a basis of $\mathfrak{sl}(2, \mathbb{R})$ to compute the Killing form $K_{\mathfrak{sl}(2,\mathbb{R})}(X, Y) = 4\mathrm{tr}(XY)$.

Exercise 3 (Complex Lie algebras)**.** Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathrm{Hom}} \mathbb{C}$ the complexification of $\mathfrak{g}$ as a vector space.

(1) Show that the bracket $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ extends uniquely to a $\mathbb{C}$-bilinear map $[\cdot, \cdot]_{\mathbb{C}} \colon \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ turning $\mathfrak{g}_{\mathbb{C}}$ into a complex Lie algebra.

$$\begin{tikzcd}
\mathfrak{g} \times \mathbb{C} \arrow[r, "h"] \arrow[d, "\otimes"'] & V \\
\mathfrak{g}_{\mathbb{C}} \arrow[ur, "\exists! \bar{h}"'] &
\end{tikzcd}$$

(2) Show that the canonical injection $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$, $X \mapsto X \otimes 1$ is a homomorphism of real Lie algebras and, if we identify $\mathfrak{g}$ with its image in $\mathfrak{g}_{\mathbb{C}}$, we have that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}.$$

Express the bracket of $\mathfrak{g}_{\mathbb{C}}$ in this decomposition.

(3) Show that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_{\mathbb{C}}$ is solvable,

(4) Show that $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}_{\mathbb{C}}$ is nilpotent.

Exercise 4 (Weight spaces and ideals)**.** Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal and let $\pi \colon \mathfrak{g} \to \mathfrak{gl}(V)$ a finite-dimensional complex representation. For a given linear functional $\lambda \colon \mathfrak{h} \to \mathbb{C}$ consider its weight space

$$V_\lambda^{\mathfrak{h}} \coloneqq \{v \in V \mid \pi(X)v = \lambda(X)v \quad \forall X \in \mathfrak{h}\}.$$

Show that every weight space $V_\lambda^{\mathfrak{h}}$ is invariant under $\pi(\mathfrak{g})$, i.e. $\pi(Y)V_\lambda^{\mathfrak{h}} \subseteq V_\lambda^{\mathfrak{h}}$ for every $\lambda \in \mathfrak{h}^*, Y \in \mathfrak{g}$.

Exercise 5 (Lie's theorem for Lie algebras)**.** Let $\mathfrak{g}$ be a solvable Lie algebra and let $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional complex representation.

Show that $\rho(\mathfrak{g})$ stabilizes a flag $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$, with $\mathrm{codim} V_i = i$, i.e. $\rho(X)V_i \subseteq V_i$ for every $X \in V_i$, $i = 1, \ldots, n$.

<u>Hint:</u> Use exercise 4.