Exercise Sheet 1

Exercise 1 (Compact-Open Topology)**.** Let X, Y, Z be a topological space, and denote by $C(Y, X) :=$ $\{f : Y \to X \text{ continuous}\}$ the set of continuous maps from Y to X. The set $C(Y, X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K, U) := \{f \in C(Y, X) \,|\, f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Prove the following useful facts about the compact-open topology.

If Y is locally compact[1] , then:

a) The evaluation map $e : C(Y, X) \times Y \to X, e(f, y) := f(y)$, is continuous.

b) A map $f : Y \times Z \to X$ is continuous if and only if the map

$$\hat{f} : Z \to C(Y, X), \quad \hat{f}(z)(y) = f(y, z),$$

is continuous.

Solution. a) For $(f, y) \in C(Y, X) \times Y$ let $U \subset X$ be an open neighborhood of $f(y)$. Since Y is locally compact, continuity of f implies there is a compact neighborhood $K \subset Y$ of y such that $f(K) \subset U$. Then $S(K, U) \times K$ is a neighborhood of (f, y) in $C(Y, X) \times Y$ taken to U by e, so e is continuous at (f, y).

b) Suppose $f : Y \times Z \to X$ is continuous. To show continuity of $\hat{f}$ it suffices to show that for a subbasic set $S(K, U) \subset C(Y, X)$, the set $\hat{f}^{-1}(S(K, U)) = \{z \in Z \,|\, f(K, z) \subset U\}$ is open in Z. Let $z \in \hat{f}^{-1}(S(K, U))$. Since $f^{-1}(U)$ is an open neighborhood of the compact set $K \times \{z\}$, there exist open sets $V \subset Y$ and $W \subset Z$ whose product $V \times W$ satisfies $K \times \{z\} \subset V \times W \subset f^{-1}(U)$. Indeed, $f^{-1}(U) = \cup_{i \in I} V_i \times W_i$ and we can choose a finite family $I' \subset I$ with $K \times \{z\} \subset \cup_{i \in I'} V_i \times W_i$. Then set $W := \cap_{z \in W_i} W_i$ and $V := \cup_{z \in W_i} V_i$.

So W is a neighborhood of z in $\hat{f}^{-1}(S(K, U))$. (The hypothesis that Y is locally compact is not needed here.)

For the converse of b) note that f is the composition $Y \times Z \to Y \times C(Y, X) \to X$ of $\mathrm{Id} \times \hat{f}$ and the evaluation map, so part a) gives the result.

Exercise 2 (General Linear Group $\mathrm{GL}(n, \mathbb{R})$)**.** The general linear group

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n \times n} \,|\, \det A \neq 0\} \subseteq \mathbb{R}^{n \times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$j : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Homeo}(\mathbb{R}^n),$$
$$A \mapsto (x \mapsto Ax).$$

[1] A subset $C \subseteq Y$ that contains an open subset $U \subseteq Y$ with $y \in U \subseteq C \subseteq Y$ is called a *neighborhood of* $y \in Y$. Then Y is called *locally compact* if for every $y \in Y$ there is a set $\mathcal{D}$ of compact neighborhoods of y such that every neighborhood of y contains an element of $\mathcal{D}$ as a subset.

a) Show that $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset C(\mathbb{R}^n,\mathbb{R}^n)$ is endowed with the compact-open topology.

Solution. Note that

$$j(\mathrm{GL}(n,\mathbb{R})) = \{f \in \mathrm{Homeo}(\mathbb{R}^n) : f(\lambda x + y) = \lambda f(x) + f(y) \text{ for all } \lambda \in \mathbb{R}, x, y \in \mathbb{R}^n\}.$$

Since evaluation is continuous also the maps

$$F_{\lambda,x,y} : \mathrm{Homeo}(\mathbb{R}^n) \to \mathbb{R}^n$$
$$f \mapsto f(\lambda x + y) - \lambda f(x) - f(y)$$

are continuous for all $\lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$.

Thus,

$$j(\mathrm{GL}(n,\mathbb{R})) = \bigcap_{\lambda \in \mathbb{R}, x,y \in X} F^{-1}_{\lambda,x,y}(0) \subset \mathrm{Homeo}(\mathbb{R}^n)$$

is closed as the intersection of closed sets.

b) If we identify $\mathrm{GL}(n,\mathbb{R})$ with its image $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with the usual topology coming from the inclusion $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$.
Hint: Exercise 1 can be useful here.

Solution. Consider the inclusions

$$i : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n},$$
$$A \mapsto \begin{pmatrix} | & & | \\ A\mathbf{e}_1 & \cdots & A\mathbf{e}_n \\ | & & | \end{pmatrix},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denotes the standard basis of $\mathbb{R}^{n\times n}$.

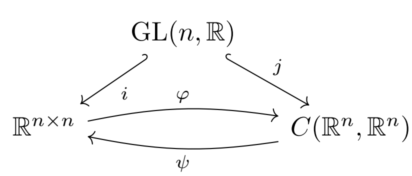
Further, consider the maps

$$\varphi : \mathbb{R}^{n\times n} \to C(\mathbb{R}^n, \mathbb{R}^n),$$
$$\begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} \mapsto (\mathbf{x} \mapsto x_1 \cdot \mathbf{v}_1 + \cdots + x_n \cdot \mathbf{v}_n),$$

and

$$\psi : C(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^{n\times n},$$
$$f \mapsto \begin{pmatrix} | & & | \\ f(\mathbf{e}_1) & \cdots & f(\mathbf{e}_n) \\ | & & | \end{pmatrix}.$$

It is easy to verify that these form the following commutative diagram.



Since both topologies under consideration on $\mathrm{GL}(n,\mathbb{R})$ come from pulling back the topologies of $\mathbb{R}^{n\times n}$ resp. $C(\mathbb{R}^n,\mathbb{R}^n)$ via i resp. j they will coincide if we can show that the maps φ and ψ are continuous[2].

[2] Let τ_i, τ_j denote the topologies, so that τ_i is the smallest topology on $\mathrm{GL}(n,\mathbb{R})$ such that i is continuous and τ_j is the smallest such that j is continuous. If φ is continuous, then

$$j = \varphi \circ i : (\mathrm{GL}(n,\mathbb{R}), \tau_i) \to C(\mathbb{R}^n, \mathbb{R}^n)$$

is continuous, thus $\tau_j \subset \tau_i$. Analogously, if ψ is continuous, then $\tau_i \subset \tau_j$ and so the two topologies coincide.

The map ψ is continuous because it is the product of the evaluation maps

$$\mathrm{ev}_{\mathbf{e}_i} : C(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n, \mathrm{ev}_{\mathbf{e}_i}(f) = f(\mathbf{e}_i)$$

$(i = 1, \dots, n)$.

Further, observe that the map

$$\mathrm{ev} \circ (\varphi \times \mathrm{Id}) : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n, (A, x) \mapsto Ax$$

is continuous. This implies that φ is continuous.

Exercise 3 ($\mathrm{O}(p,q)$)**.** We consider the orthogonal group $\mathrm{O}(p,q)$ of signature $p, q \geq 1$.

a) Show that the connected component of the group $O(1,1)$ containing the identity is homeomorphic to $\mathbb{R}$.

b) Show that for all $p, q \geq 1$, $\mathrm{O}(p,q)$ has a subgroup isomorphic to $\mathbb{R}$.

Solution. a) We recall that

$$\mathrm{O}(1,1) = \left\{ g \in \mathrm{GL}(2,\mathbb{R}) \colon {}^{\mathrm{t}}g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Now if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{O}(1,1),$$

we obtain the conditions $a^2 - c^2 = 1, d^2 - b^2 = 1$ and $ab = cd$. Rephrasing $a = cd/b$ and $b = cd/a$ we obtain

$$\begin{aligned} & \left(\frac{cd}{b}\right)^2 - c^2 = 1 \quad \text{and} \quad d^2 - \left(\frac{cd}{a}\right)^2 = 1 \\ \iff & c^2d^2 - c^2b^2 = b^2 \quad \text{and} \quad a^2d^2 - c^2d^2 = a^2 \\ \iff & 1 = d^2 - b^2 = b^2/c^2 \quad \text{and} \quad 1 = a^2 - c^2 = a^2/d^2 \\ \iff & b^2 = c^2 \quad \text{and} \quad a^2 = d^2, \end{aligned}$$

so $a = \pm d$ and $b = \pm c$. By $ab = cd$, both signs have to be the same. We obtain that

$$\mathrm{O}(1,1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \colon a^2 - b^2 = 1 \right\} \cup \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \colon a^2 - b^2 = 1 \right\}.$$

Every a, b with $a^2 - b^2 = 1$ can be written as $a = \cosh(\varphi)$ and $b = \sinh(\varphi)$ for some $\varphi \in \mathbb{R}$. The description of $\mathrm{O}(1,1)$ above shows that there are two parts of $\mathrm{O}(1,1)$, both of which are pathconnected, (parametrize the paths using φ). We claim that the two parts are distinct connected components:

Note that the determinant on the first part is $a^2 - b^2 = 1$ and the determinant on the second part is $-a^2 + b^2 = -1$. Since the determinant is a continuous map $\mathrm{O}(1,1) \to \mathbb{R}$ whose image is not connected, also $O(1,1)$ is not connected.

We note that the first component contains Id when $\varphi = 0$, so the connected component of the identity is

$$\mathrm{O}(1,1)^\circ = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \colon a^2 - b^2 = 1 \right\} = \left\{ \begin{pmatrix} \cosh(\varphi) & \sinh(\varphi) \\ \sinh(\varphi) & \cosh(\varphi) \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \colon \varphi \in \mathbb{R} \right\}$$

and the last description shows that it is homeomorphic to $\mathbb{R}$.

b) Equipped with the ideas from part a), we consider the subgroup

$$G = \{g(\varphi)\colon \varphi \in \mathbb{R}\} \quad \text{for} \quad g(\varphi) = \left(\begin{array}{ccc|ccc} \cosh(\varphi) & 0 & \cdots & \sinh(\varphi) & 0 & \cdots \\ 0 & 1 & \ddots & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \\ \hline \sinh(\varphi) & 0 & \cdots & \cosh(\varphi) & 0 & \cdots \\ 0 & 0 & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots \end{array}\right)$$

Explicit calculations show that $g(\varphi) \in \mathrm{O}(p,q)$.

Exercise 4 (Isometry Group $\mathrm{Iso}(X)$)**.** Let (X,d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x,y) \quad \text{for all } x, y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem, see Appendix A.2 in Prof. Alessandra Iozzi's book.

Solution. The compact-open topology on $\mathrm{Homeo}(X)$ coincides with the topology induced by the metric of uniform convergence

$$d_\infty(f,g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

Note that by Arzelà–Ascoli (Theorem A.1 in the lecture notes) a family $\mathcal{F} \subseteq C(X,X)$ of continuous maps is compact if and only if $\mathcal{F}$ is equicontinuous, and $\mathcal{F}$ is closed.

Equicontinuity of $\mathcal{F} := \mathrm{Iso}(X)$ is clear, because we are dealing with isometries. We check that $\mathrm{Iso}(X)$ is closed.

Let $f \in C(X,X)$ and let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Iso}(X)$ be a sequence converging to it. Let $x, y \in X$ then

$$\begin{aligned} 0 &\leq |d(f(x), f(y)) - d(x,y)| \\ &= |d(f(x), f(y)) - d(f_n(x), f_n(y))| \\ &\leq |d(f(x), f(y)) - d(f_n(x), f(y))| + |d(f_n(x), f(y)) - d(f_n(x), f_n(y))| \\ &\leq d(f(x), f_n(x)) + d(f(y), f_n(y)) \to 0 \quad (n \to \infty). \end{aligned}$$

Hence, f is an isometry as wished for. Because f was arbitrary this shows that $\mathrm{Iso}(X) \subseteq C(X,X)$ is closed.

Exercise† 5 (Homeomorphism Group $\mathrm{Homeo}(X)$)**.**

a) Let X be a *compact* Hausdorff space. Show that $(\mathrm{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.

Solution. Denote by $m : \mathrm{Homeo}(X) \times \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ the composition $m(f,g) = f \circ g$ and by $i : \mathrm{Homeo}(X) \to \mathrm{Homeo}(X)$ the inversion $i(f) = f^{-1}$. We need to see that m and i are continuous.

i) m is continuous: We want to show that m is continuous at any tuple $(f,g) \in \mathrm{Homeo}(X) \times \mathrm{Homeo}(X)$. Thus let $S(K,U) \ni f \circ g$ be a subbasis neighborhood of $f \circ g$, i.e. $K \subset X$ is compact and $U \subset X$ is open such that $f(g(K)) \subset U$. Observe that $g(K)$ is compact and is contained in $f^{-1}(U)$ which is open. Because X is (locally) compact we may find an open set $V \subset X$ with compact closure $\overline{V}$ such that

$$g(K) \subset V \subset \overline{V} \subset f^{-1}(U).$$

It is now easy to verify that $W := S(\overline{V}, U) \times S(K, V)$ is an open neighborhood of (f, g) such that $m(W) \subset S(K, U)$. Indeed, (f, g) is by construction of V contained in W and for any $(h_1, h_2) \in W$ we get

$$h_2(K) \subset V \subset \overline{V} \subset h_1^{-1}(U).$$

Hence, m is continuous at every point of $\operatorname{Homeo}(X) \times \operatorname{Homeo}(X)$.

ii) i is continuous: Let $f \in \operatorname{Homeo}(X)$, $K \subset X$ compact and $U \subset X$ open. Then

$$\begin{aligned} i(f) \in S(K,U) &\iff f^{-1}(K) \subset U \iff K \subset f(U) \\ &\iff f(U^c) = f(U)^c \subset K^c \iff f \in S(U^c, K^c). \end{aligned}$$

Observe that U^c is compact as a closed subset of the compact space X and that K^c is open as the complement of a (compact) closed set.

This shows that $i^{-1}(S(K,U)) = S(U^c, K^c)$ for every element $S(K,U)$ of a subbasis for the compact-open topology on $\operatorname{Homeo}(X)$, whence i is continuous.

b) The objective of this exercise is to show that $(\operatorname{Homeo}(X), \circ)$ will not necessarily be a topological group if X is only locally compact.

Consider the "middle thirds" Cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1]$$

in the unit interval. We define the sets $U_n = C \cap [0, 3^{-n}]$ and $V_n = C \cap [1-3^{-n}, 1]$. Further we construct a sequence of homeomorphisms $h_n \in \operatorname{Homeo}(C)$ as follows:

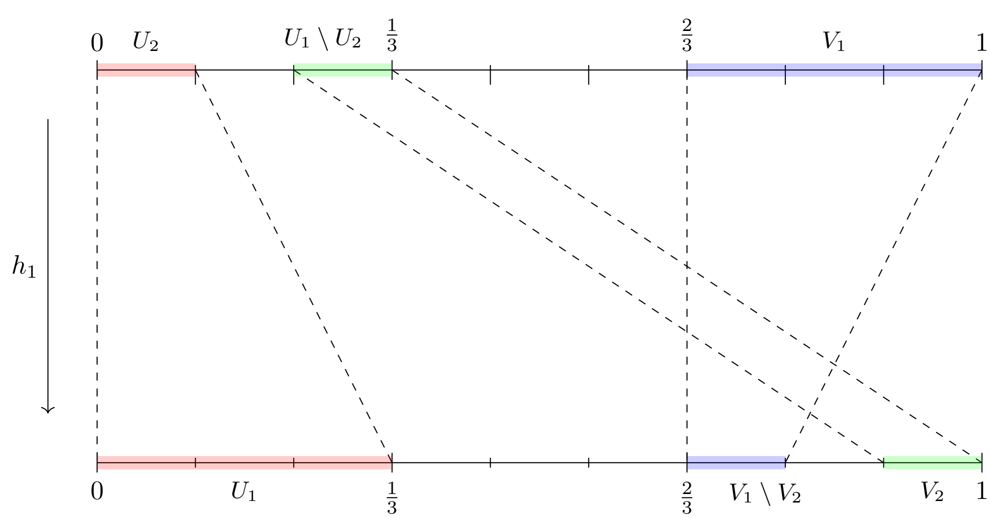
- $h_n(x) = x$ for all $x \in C \setminus (U_n \cup V_n)$,
- $h_n(0) = 0$,
- $h_n(U_{n+1}) = U_n$,
- $h_n(U_n \setminus U_{n+1}) = V_{n+1}$,
- $h_n(V_n) = V_n \setminus V_{n+1}$.

These restrict to homeomorphisms $h_n|_X$ on $X := C \setminus \{0\}$.

Show that the sequence $(h_n|_X)_{n\in\mathbb{N}} \subset \operatorname{Homeo}(X)$ converges to the identity on X but the sequence $((h_n|_X)^{-1})_{n\in\mathbb{N}} \subset \operatorname{Homeo}(X)$ of their inverses does not!

Remark: However, if X is locally compact and *locally connected* then $\operatorname{Homeo}(X)$ is a topological group.

Solution. The following picture gives a pictorial description of what h_1 does on the Cantor set C.



Since $h_n(0) = 0$ we obtain indeed a homeomorphism $h_n|_X \in \text{Homeo}(X)$ by restriction to $X = C \setminus \{0\}$. Let us first see that the sequence $(h_n|_X)_{n\in\mathbb{N}}$ indeed converges to $\text{Id} \in \text{Homeo}(X)$. For that let $S(K,U)$ be a subbasis neighborhood of Id, i.e. K is a compact subset of X contained in some open set $U \subset X$. Therefore we can find an $M \in \mathbb{N}$ such that U_M and K are disjoint.

If $1 \notin K$ then there is also an $N \geq M$ such that V_n and K are disjoint. In this case $h_n|K$ is the identity and hence in $S(K,U)$ for all $n \geq N$.

If $1 \in K$ then there is an $N \geq M$ such that V_N is contained in U. Consequently, we have

$$h_n(K \setminus V_n) = K \setminus V_n, \quad h_n(K \cap V_n) \subset V_n \subset V_N \subset U,$$

for all $n \geq N$.

In any case the sequence $(h_n|_X)_{n\in\mathbb{N}}$ will be in $S(K,U)$ for large enough n such that $\lim_{n\to\infty} h_n|_X = \text{Id}$. On the other hand $h_n^{-1}(1) \in U_n$ for every $n \in \mathbb{N}$ such that $\lim_{n\to\infty} h_n^{-1}(1) = 0$. Thus the sequence $(h_n^{-1}|_X)_{n\in\mathbb{N}}$ certainly does not converge to Id.

Remark: Note that we actually needed to remove 0 from C for this construction to work. In fact, the sequence h_n does not converge to Id in $\text{Homeo}(C)$:

Let $K = [0, 1/9] \cap C, U = [0, 1/2) \cap C$. Then $S(K,U)$ is again a neighborhood of Id. However, $U_n \subset K$ for every $n \geq 2$ and $V_{n+1} \subset U^c$ which implies that

$$h_n(U_n \setminus U_{n+1}) \subset U^c,$$

i.e. $h_n \notin S(K,U)$.

c) Let $\mathbb{S}^1 \subset \mathbb{C} \setminus \{0\}$ denote the circle. Show that $\text{Homeo}(\mathbb{S}^1)$ is not locally compact.

Remark: In fact, $\text{Homeo}(M)$ is not locally compact for any manifold M.

Solution. We will prove a more general fact, namely that $\text{Homeo}(M)$ is not locally compact for any compact manifold M. Note that we can think of M as a compact metric space (M,d) by Urysohn's metrization theorem. In the case when M is a smooth manifolds this is even easier to see by endowing it with a Riemannian metric. This puts us now in the favorable position of being able to identify the compact-open topology on $\text{Homeo}(X)$ with the topology of uniform convergence.

We denote by

$$d_\infty(f,g) := \sup\{d(f(x), g(x)) : x \in M\}$$

the metric of uniform convergence on $\text{Homeo}(M)$. Further denote by $B_f^\infty(r)$ the ball of radius $r > 0$ about a homeomorphism $f \in \text{Homeo}(M)$. In order to show that $\text{Homeo}(M)$ is not locally compact we will construct in every $\varepsilon > 0$ ball about the identity $B_{\text{Id}}^\infty(\varepsilon)$ a sequence of homeomorphisms $(f_k)_{k\in\mathbb{N}}$ with no convergent subsequence.

Let $\varepsilon > 0$ and denote $B = B_{\text{Id}}^\infty(\varepsilon)$. Further, let $x_0 \in M$ and choose a coordinate chart $\varphi : U \subset B_{\varepsilon/2}(x_0) \to \mathbb{R}^n$ centered at x_0 (i.e. $\varphi(x_0) = 0$) contained in the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ about x_0 in M. Consider the homeomorphisms

$$\psi_k : \overline{B_1}(0) \to \overline{B_1}(0), x \mapsto \|x\|^k x$$

on the closed unit ball $\overline{B_1}(0)$ in $\mathbb{R}^n$ which fix $0 \in \mathbb{R}^n$ and the boundary n-sphere pointwise. Note that the sequence $(\psi_k)_{k\in\mathbb{N}}$ converges pointwise to

$$\psi_\infty = \begin{cases} x, & \text{if } x \in \partial B_1(0), \\ 0, & \text{if } x \in B_1(0). \end{cases}$$

Now, define

$$f_k(x) := \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ \varphi^{-1}(\psi_k(\varphi(x))), & \text{if } x \in \varphi^{-1}(B_1(0)). \end{cases}$$

It is easy to see that the maps $f_k : M \to M$ are indeed homeomorphisms: $f_k|_{\varphi^{-1}(\overline{B_1}(0))^c} = \text{Id} : \varphi^{-1}(\overline{B_1}(0))^c \to \varphi^{-1}(\overline{B_1}(0))^c$ is a homeomorphism, $\varphi^{-1} \circ \psi_k \circ \varphi : \varphi^{-1}(\overline{B_1}(0)) \to \varphi^{-1}(\overline{B_1}(0))$ is a homeomorphism and both coincide on $\varphi^{-1}(\partial B_1(0))$.

Further, the homeomorphisms f_k map the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ to itself and fix x_0. Therefore,

$$d(f_k(x), x) \leq d(f_k(x), \underbrace{f_k(x_0)}_{=x_0}) + d(x_0, x) < \varepsilon,$$

for every $x \in B_{\varepsilon/2}(x_0)$, and clearly $f_k(x) = x$ for every $x \notin B_{\varepsilon/2}(x_0)$. Hence, the sequence $(f_k)_{k\in\mathbb{N}}$ is in $B_\varepsilon^\infty(\mathrm{Id})$.

However, the sequence $(f_k)_{k\in\mathbb{N}}$ converges pointwise to

$$f_\infty(x) = \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ x_0, & \text{if } x \in \varphi^{-1}(B_1(0)), \end{cases}$$

If there were a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ converging to some $f \in \mathrm{Homeo}(M)$ uniformly then this sequence would also converge pointwise to f, i.e. f needs to coincide with f_∞. But f_∞ is not even continuous which contradicts our assumption of $f \in \mathrm{Homeo}(M)$. Therefore $(f_k)_{k\in\mathbb{N}} \subset B_\varepsilon^\infty(\mathrm{Id})$ has no uniformly convergent subsequences.

Exercise 6. Locally Compact Hilbert Spaces are Finite-Dimensional Let $\mathcal{H}$ be a Hilbert space a field $k = \mathbb{R}$ or $\mathbb{C}$. Show that $\mathcal{H}$ is locally compact if and only if it is finite-dimensional.

Solution. One direction is easy: If $\mathcal{H}$ is finite-dimensional it is isomorphic to k^n which is locally compact by Heine–Borel.

Suppose $\mathcal{H}$ is infinite-dimensional and locally compact. Then every finite-dimensional subspace $V \subseteq \mathcal{H}$ is proper. In particular, its orthogonal complement $V^\perp$ is non-empty. We will now construct a sequence of vectors $(w_n)_{n\in\mathbb{N}}$ inductively. Let us start with a unit vector $w_1 \in \mathcal{H}, \|w_1\| = 1$, and set $V_1 := \langle v_1 \rangle$. We choose $w_{k+1} \in \mathcal{H}, \|w_{k+1}\| = 1$, inductively as a unit vector in $V_k^\perp$ and set $V_{k+1} := \langle w_1, \ldots, w_{k+1} \rangle$. Note that $\|w_k - w_l\| = 1$ for all $1 \leq l < k$ by definition. Therefore, $(w_k)_{k\in\mathbb{N}}$ admits no convergent subsequence.

On the other hand, there is some $r > 0$ such that the closed ball $\overline{B}_r(0) \subset \mathcal{H}$ is compact, because $\mathcal{H}$ is locally compact. Rescaling this ball shows that any closed ball in $\mathcal{H}$ is compact, in particular the closed unit ball is compact, too. The sequence $(w_k)_{k\in\mathbb{N}}$ is contained in the closed unit ball and admits a convergent subsequence by compactness; contradiction!

Exercise 7 (Unitary Operators)**.** Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H})$ its group of unitary operators. Show that the weak operator topology coincides with the strong operator topology on $U(\mathcal{H})$.

Hint: Recall that a sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *weak operator topology* if

$$\lambda(T_n x) \to \lambda(Tx) \quad (n \to \infty)$$

for every linear functional $\lambda \in \mathcal{H}^*$ and every $x \in \mathcal{H}$.

A sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *strong operator topology* if

$$T_n x \to Tx \quad (n \to \infty)$$

for every $x \in \mathcal{H}$.

Solution. Recall that a sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *weak operator topology* if

$$\lambda(T_n x) \to \lambda(Tx) \quad (n \to \infty)$$

for every linear functional $\lambda \in \mathcal{H}^*$ and every $x \in \mathcal{H}$.

A sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *strong operator topology* if

$$T_n x \to Tx \quad (n \to \infty)$$

for every $x \in \mathcal{H}$.

In order to show that the weak operator topology coincides with the strong operator topology it will be sufficient to show that a sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ converges with respect to the weak operator topology to $T \in U(\mathcal{H})$ if and only if $(T_n)_{n\in\mathbb{N}}$ converges with respect to the strong operator topology to T.

" $\Longleftarrow$ ": Let $T_n \to T$ strongly and let $\lambda \in \mathcal{H}^*, x \in \mathcal{H}$. Then because λ is continuous and $T_n x \to T_x$ we get

$$\lambda(T_n x) \to \lambda(Tx)$$

as $n \to \infty$.

" $\Longrightarrow$ ": Let $T_n \to T$ weakly and let $x \in \mathcal{H}$. We need to see that

$$\|T_n x - Tx\|^2 \to 0 \quad (n \to \infty).$$

We compute

$$\begin{aligned}
\|T_n x - Tx\|^2 &= \langle T_n x - Tx, T_n - Tx\rangle \\
&= \langle T_n x, T_n x\rangle - \langle T_n x, Tx\rangle - \langle Tx, T_n x\rangle + \langle Tx, Tx\rangle \\
&= \langle x, x\rangle - \langle T_n x, Tx\rangle - \langle Tx, T_n x\rangle + \langle x, x\rangle \\
&= 2\|x\|^2 - \left(\langle T_n x, Tx\rangle + \overline{\langle T_n x, Tx\rangle}\right) \\
&= 2\|x\|^2 - 2\Re\left(\langle T_n x, Tx\rangle\right) \\
&\to 2\|x\|^2 - 2\|Tx\|^2 = 2\|x\|^2 - 2\|x\|^2 = 0 \quad (n \to \infty),
\end{aligned}$$

where we have used that T_n and T are unitary and that $\langle \cdot, Tx\rangle$ is a continuous linear functional.

Exercise 8 (p-adic Integers $\mathbb{Z}_p$)**.** Let $p \in \mathbb{N}$ be a prime number. Recall that the p-adic integers $\mathbb{Z}_p$ can be seen as the subspace

$$\left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} : a_{n+1} \equiv a_n \pmod{p^n}\right\}$$

of the infinite product $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}_p$ carrying the induced topology. Note that each $\mathbb{Z}/p^n\mathbb{Z}$ carries the discrete topology and $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is endowed with the resulting product topology.

a) Show that the image of $\mathbb{Z}$ via the embedding

$$\begin{aligned}
\iota : \mathbb{Z} &\to \mathbb{Z}_p, \\
x &\mapsto (x \pmod{p^n})_{n\in\mathbb{N}}
\end{aligned}$$

is dense. In particular, $\mathbb{Z}_p$ is a compactification of $\mathbb{Z}$.

Solution. Let $(x_n) \in \mathbb{Z}_p$. A neighborhood basis of (x_n) is given by the sets

$$B_m((x_n)) = \{(y_n) \in \mathbb{Z}_p : x_1 = y_1, \ldots, x_m = y_m\}, \quad m \in \mathbb{N}.$$

Let $m \in \mathbb{N}$. We want to construct an integer $x \in \mathbb{Z}$ such that $\iota(x) \in B_m((x_n))$. It suffices to take a preimage $x \in \mathbb{Z}$ of $x_m \in \mathbb{Z}/p^m\mathbb{Z}$ under $\pi_m : \mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$. Then we clearly obtain

$$\begin{aligned}
x_m &\equiv x (\text{mod } p^m), \\
x_{m-1} &\equiv x_m (\text{mod } p^{m-1}) \equiv x (\text{mod } p^{m-1}), \\
&\vdots \\
x_1 &\equiv x (\text{mod } p).
\end{aligned}$$

That is $\iota(x) \in B_m((x_n))$.

b) Show that the 2-adic integers $\mathbb{Z}_2$ are homeomorphic to the "middle thirds" cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1].$$

Solution. We will prove that the map

$$\varphi : C \to \mathbb{Z}_2,$$
$$\sum_{n=1}^{\infty} \varepsilon_n 3^{-n} \mapsto \left(\sum_{k=1}^{n} \frac{\varepsilon_k}{2} \cdot 2^{k-1} \right)_{n\in\mathbb{N}}$$

is a homeomorphism.

φ is well-defined because

$$\varphi\left(\sum_{n=1}^{\infty} \varepsilon_n 3^{-n} \right)_n \equiv \sum_{k=1}^{n} \frac{\varepsilon_k}{2} \cdot 2^{k-1} + \frac{\varepsilon_{n+1}}{2} \cdot 2^n \equiv \varphi\left(\sum_{n=1}^{\infty} \varepsilon_n 3^{-n} \right)_{n+1} \pmod{2^n}.$$

By uniqueness of 2-adic expansions φ is injective.

φ is surjective because for every $(x_n)_{n\in\mathbb{N}} \in \mathbb{Z}_2$ we can find 2-adic expansions

$$x_n = a_0^{(n)} + a_1^{(n)} \cdot 2 + \cdots + a_{n-1}^{(n)} \cdot 2^{n-1}, \quad n \in \mathbb{N},$$

with unique $a_i^{(n)} \in \{0,1\}$. By the compatibility condition in $\mathbb{Z}_2$

$$x_n \equiv x_{n+1} \pmod{2^n}$$

we get that $a_i^{(n)} = a_i^{(n+1)}$ for every $i < n$. Hence, we can write

$$x_n = a_0 + a_1 \cdot 2 + \cdots + a_{n-1} \cdot 2^{n-1}, \quad n \in \mathbb{N},$$

with unique $a_i \in \{0,1\}$. Thus,

$$\varphi\left(\sum_{n=1}^{\infty} 2a_n 3^{-n} \right) = (x_n)_{n\in\mathbb{N}},$$

i.e. φ is surjective.

In order to prove that φ is continuous and open we first need to deduce the following neat relation: For every $c = \sum_{n=1}^{\infty} \varepsilon_n 3^{-n}, d = \sum_{n=1}^{\infty} \delta_n 3^{-n} \in C$

$$-\log_3 |d-c| \leq \min\{k \in \mathbb{N} : \varepsilon_k \neq \delta_k\} \leq -\log_3 |d-c| + 1.$$

Indeed, let $m = \min\{k \in \mathbb{N} : \varepsilon_k \neq \delta_k\}$. Then

$$\begin{aligned}
|d-c| &= \left| (\delta_m - \varepsilon_m) \cdot 3^{-m} + \sum_{n=m+1}^{\infty} (\delta_n - \varepsilon_n) \cdot 3^{-n} \right| \\
&\geq \left| \underbrace{|\delta_m - \varepsilon_m|}_{=2} \cdot 3^{-m} - \left| \sum_{n=m+1}^{\infty} (\delta_n - \varepsilon_n) \cdot 3^{-n} \right| \right| \\
&\geq \frac{2}{3^m} - \sum_{n=m+1}^{\infty} |\delta_n - \varepsilon_n| \cdot 3^{-n} \\
&\geq \frac{2}{3^m} - \sum_{n=m+1}^{\infty} 2 \cdot 3^{-n} = \frac{2}{3^m} - \frac{1}{3^m} = 3^{-m}.
\end{aligned}$$

Applying the logarithm to base 3 on both sides yields the first inequality.

The second inequality follows from the following easier computation.

$$\begin{aligned}|d-c| &= \left|\sum_{n=m}^{\infty}(\delta_n-\varepsilon_n)\cdot 3^{-n}\right| \leq \sum_{n=m}^{\infty} 2\cdot 3^{-n} = \frac{1}{3^{m-1}} \\ &\implies \log_3|d-c| \leq -m+1.\end{aligned}$$

Now, let $c=\sum_{n=1}^{\infty}\varepsilon_n 3^{-n}\in C$ and consider a neighborhood $B_m(\varphi(c))$. Then

$$\begin{aligned} & d=\sum_{n=1}^{\infty}\delta_n 3^{-n}\in\varphi^{-1}(B_m(\varphi(c))) \\ \iff & \sum_{k=1}^{l}\frac{\varepsilon_k}{2}\cdot 2^{k-1}=\sum_{k=1}^{l}\frac{\delta_k}{2}\cdot 2^{k-1}, \quad \forall 1\leq l\leq m \\ \iff & \varepsilon_k=\delta_k, \quad \forall k=1,\dots,m \\ \iff & \min\{k\in\mathbb{N}:\varepsilon_k\neq\delta_k\}\geq m+1\end{aligned}$$

By the previously deduced relation this readily implies

$$B_{m+1}(\varphi(c))\subset\varphi(C\cap(-3^{-m}+c,c+3^{-m}))\subset B_m(\varphi(c)).$$

It follows that φ is continuous and open.