Solutions of Exercise Sheet 4

Exercise 1 (One- and two-dimensional Lie Algebras)**.** Classify the one- and two-dimensional real Lie algebras up to Lie algebra isomorphism and realize them as Lie subalgebras of some $\mathfrak{gl}_n\mathbb{R} = \mathfrak{gl}(\mathbb{R}^n)$.

Hint: In dimension two one can show that if the Lie algebra is non-abelian then there is a basis X, Y such that $[X, Y] = Y$.

Solution. Let $(\mathfrak{a}, [\cdot, \cdot])$ be a real Lie algebra.

We will first deal with the one-dimensional case. Suppose $\dim \mathfrak{a} = 1$ and let X be a basis vector for $\mathfrak{a}$. Due to the anti-symmetry of the Lie bracket we have

$$[X, X] = -[X, X] = 0,$$

i.e. every one-dimensional Lie algebra is abelian. We claim that the linear map $\varphi : (\mathfrak{a}, [\cdot, \cdot]) \to (\mathbb{R}, [\cdot, \cdot])$ given by $\varphi(X) = 1$ is a Lie algebra isomorphism where the Lie bracket on $\mathbb{R}$ vanishes everywhere. Clearly, φ is an isomorphism of vector spaces and

$$[\varphi(X), \varphi(X)] = 0 = \varphi(\underbrace{[X, X]}_{=0})$$

such that φ is indeed a Lie algebra isomorphism.

In order to realize $\mathfrak{a}$ as a Lie subalgebra of some $\mathfrak{gl}_n\mathbb{R}$ we need to find a one-dimensional subalgebra of some $\mathfrak{gl}_n\mathbb{R}$ on which the commutator $[\cdot, \cdot]$ in $\mathfrak{gl}_n\mathbb{R}$ vanishes. Consider

$$\mathfrak{b} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \subseteq \mathfrak{gl}_2\mathbb{R}.$$

Clearly, $\mathfrak{b}$ is a linear subspace of $\mathfrak{gl}_2\mathbb{R}$. Further, note that

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x \cdot y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

for all $x, y \in \mathbb{R}$, such that $[X, Y] = 0$ for all $X, Y \in \mathfrak{b}$. Therefore the vector space isomorphism $\psi : \mathbb{R} \to \mathfrak{b}$ given by

$$\psi(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

is also a Lie algebra isomorphism. Thus, $\psi \circ \varphi : \mathfrak{a} \hookrightarrow \mathfrak{gl}_2(\mathbb{R})$ realizes $\mathfrak{a}$ as a Lie subalgebra of $\mathfrak{gl}_2\mathbb{R}$.

Suppose $\dim \mathfrak{a} = 2$ and let $\{X, Y\}$ be a basis of $\mathfrak{a}$. Suppose $\mathfrak{a}$ is abelian, i.e. $[X, Y] = 0$. Consider

$$\mathfrak{c} := \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset \mathfrak{gl}_2\mathbb{R}$$

and the vector space isomorphism $\varphi : \mathfrak{a} \to \mathfrak{c}$ given by

$$\varphi(X) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: E_{11}, \quad \varphi(Y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: E_{22}.$$

Note that

$$E_{11} \cdot E_{22} = 0 = E_{22} \cdot E_{11},$$

such that

$$\varphi([X, Y]) = \varphi(0) = 0 = [E_{11}, E_{22}] = [\varphi(X), \varphi(Y)].$$

Therefore, $\varphi : \mathfrak{a} \to \mathfrak{c} \subset \mathfrak{gl}_2\mathbb{R}$ is a Lie algebra isomorphism. This realizes $\mathfrak{a}$ as the subalgebra $\mathfrak{c}$ of $\mathfrak{gl}_2\mathbb{R}$ and shows that every real abelian Lie algebra is isomorphic to $\mathfrak{c}$.

Finally, suppose that $\mathfrak{a}$ is non-abelian such that

$$[X, Y] = \alpha X + \beta Y \neq 0 \tag{$\star$}$$

for some $\alpha, \beta \in \mathbb{R}$. By $(\star)$ not both α and β are zero such that

$$\beta\lambda - \alpha\mu = 1$$

for some $\lambda, \mu \in \mathbb{R}$. Define

$$X' := \lambda X + \mu Y, \quad Y' := \alpha X + \beta Y = [X, Y].$$

Observe that the base change from $\{X, Y\}$ to $\{X', Y'\}$ is given by the matrix

$$\begin{pmatrix} \lambda & \alpha \\ \mu & \beta \end{pmatrix}$$

with determinant $\lambda\beta - \alpha\mu = 1$ such that $\{X', Y'\}$ is again a basis of $\mathfrak{a}$. Further,

$$\begin{aligned} [X', Y'] &= [\lambda X + \mu Y, \alpha X + \beta Y] \\ &= \lambda\beta[X, Y] + \mu\alpha[Y, X] \\ &= (\beta\lambda - \alpha\mu)[X, Y] \\ &= Y'. \end{aligned}$$

Consider the vector subspace $\mathfrak{d} \subset \mathfrak{gl}_2\mathbb{R}$ generated by the matrices

$$A := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In fact, $\mathfrak{d}$ is a Lie subalgebra:

$$\begin{aligned} [A, C] &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = C. \end{aligned}$$

This computation also shows that the linear map $\varphi : \mathfrak{a} \to \mathfrak{d}$ given by

$$\varphi(X') = A, \quad \varphi(Y') = C$$

is a Lie algebra isomorphism (it is easily seen to be an isomorphism of vector spaces). Therefore, $\mathfrak{a}$ can be realized as the subalgebra $\mathfrak{d}$ of $\mathfrak{gl}_2\mathbb{R}$. This also proves that any real, non-abelian Lie algebra $\mathfrak{a}$ is isomorphic to $\mathfrak{d}$.

<u>Remark:</u> Notice that the map $\Phi : \mathfrak{gl}_2\mathbb{R} \hookrightarrow \mathfrak{gl}_n\mathbb{R}$ given by

$$\Phi(A) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$$

is an injective Lie algebra homomorphism such that the discussed realizations of $\mathfrak{a}$ as subalgebras of $\mathfrak{gl}_2\mathbb{R}$ also amount to realizations of $\mathfrak{a}$ in any $\mathfrak{gl}_n\mathbb{R}$.

Exercise 2 (Some Lie Algebras). (a) Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank r. By the constant rank theorem we know that the level set $L = f^{-1}(q)$ is a regular submanifold of M of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_pL \cong \ker d_pf$$

for every $p \in L = f^{-1}(q)$.

<u>Hint:</u> Describe elements in T_pL as $\gamma'(0)$ for a smooth path $\gamma\colon (-\varepsilon, \varepsilon) \to M$.

Solution. Since $L = f^{-1}(q)$ is a regular submanifold of M we may think of the tangent space T_pL as a subspace of the tangent space T_pM. We will first show that $T_pL \subseteq \ker d_pf$. Let $v \in T_pL$ and let $\gamma : (-\varepsilon, \varepsilon) \to L = f^{-1}(q)$ be a smooth curve in L such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then $f(\gamma(t)) = q$ for all $t \in (-\varepsilon, \varepsilon)$, i.e. $f \circ \gamma$ is the constant curve. It follows that

$$d_pf(v) = d_{\gamma(0)}f(\gamma'(0)) = \frac{d}{dt}\bigg|_{t=0} f(\gamma(t)) = 0.$$

In particular, $v \in \ker d_pf$ as claimed.

Finally, note that $\ker d_pf$ is a subspace of T_pM of dimension

$$\dim \ker d_pf = \dim T_pM - \operatorname{rank} d_pf = \dim M - r = \dim L = \dim T_pL.$$

Therefore T_pL is a linear subspace of $\ker d_pf$ of maximal dimension such that $T_pL = \ker d_pf$.

(b) Use part a) to compute the Lie algebras of the following Lie groups: $\mathrm{O}(n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{R})$, $\mathrm{O}(p, q)$, $B(n)$ the group of real invertible upper triangular matrices and $N(n)$ the subgroup of $B(n)$ with only ones on the diagonal.

Solution. Note that all of the listed Lie groups are subgroups of $\mathrm{GL}(n, \mathbb{K})$ that are also regular submanifolds ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). In particular the inclusion maps yield injective Lie algebra homomorphisms. This implies that the corresponding Lie algebras can be canonically identified with Lie subalgebras of $\mathfrak{gl}_n\mathbb{K}$. Hence the Lie bracket will be given by the ambient Lie bracket $[\cdot, \cdot]$ of $\mathfrak{gl}_n\mathbb{K}$. Identifying $\mathfrak{gl}_n\mathbb{K} \cong T_I \,\mathrm{GL}(n, \mathbb{K}) \cong \mathbb{K}^{n\times n}$ the Lie bracket is given by the commutator

$$[A, B] = AB - BA.$$

(i) $O(n, \mathbb{R})$: Consider the function $f_1 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_1(A) = A^T A$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$. It is easy to check that f_1 has constant rank and that

$$O(n) = f_1^{-1}(I).$$

By part a)

$$\mathfrak{o}(n) := \mathrm{Lie}(O(n)) \cong T_I O(n) \cong \ker d_I f_1 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I \,\mathrm{GL}(n, \mathbb{R})$. We compute

$$\begin{aligned} d_I f_1(X) &= \frac{d}{dt}\bigg|_{t=0} (I + tX)^t(I + tX) \\ &= X^t + X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t + X = 0\}.$$

(ii) $O(p, q)$: Consider the function $f_2 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_2(A) = A^T I_{p,q} A$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$, where

$$I_{p,q} = \mathrm{diag}(\underbrace{1, \dots, 1}_{p\text{-times}}, \underbrace{-1, \dots, -1}_{q\text{-times}}).$$

It is easy to check that f_2 has constant rank and that

$$O(p, q) = f_2^{-1}(I_{p,q}).$$

By part a)

$$\mathfrak{o}(p,q) := \mathrm{Lie}(O(p,q)) \cong T_I O(p,q) \cong \ker d_I f_2 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned} d_I f_2(X) &= \left.\frac{d}{dt}\right|_{t=0} (I+tX)^t I_{p,q}(I+tX) \\ &= X^t I_{p,q} + I_{p,q}X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(p,q) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t I_{p,q} + I_{p,q}X = 0\}.$$

(iii) $B(n)$: Consider the function $f_5 : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_5(A) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ A_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{n1} & \cdots & A_{n,n-1} & 0 \end{pmatrix}$$

for every $A \in \mathrm{GL}(n,\mathbb{R})$. It is easy to check that f_5 has constant rank and that

$$B(n) = f_5^{-1}(0).$$

By part a)

$$\mathfrak{b}(n) := \mathrm{Lie}(B(n)) \cong T_I B(n) \cong \ker d_I f_5 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned} d_I f_5(X) &= \left.\frac{d}{dt}\right|_{t=0} f_5(I+tX) \\ &= \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ X_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ X_{n1} & \cdots & X_{n,n-1} & 0 \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathfrak{b}(n) = \left\{ \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ & \ddots & \vdots \\ 0 & & X_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n} \right\}.$$

(iv) $N(n)$: Consider the function $f_6 : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_6(A) = \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

for every $A \in \mathrm{GL}(n,\mathbb{R})$. It is easy to check that f_6 has constant rank and that

$$N(n) = f_6^{-1}(I).$$

By part a)

$$\mathfrak{n}(n) := \mathrm{Lie}(N(n)) \cong T_I N(n) \cong \ker d_I f_6 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned} d_I f_6(X) &= \left.\frac{d}{dt}\right|_{t=0} f_6(I+tX) \\ &= \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathfrak{n}(n) = \left\{ \begin{pmatrix} 0 & * & * \\ \vdots & \ddots & * \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n} \right\}.$$

Exercise 3 (Distributions and Lie Subalgebras)**.** a) Let M be a smooth manifold, $X, Y \in \mathrm{Vect}(M)$ vector fields on M, and $f, g \in C^\infty(M)$ smooth functions. Show that

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

Solution. Let $h \in C^\infty(M)$ and $p \in M$. We compute

$$\begin{aligned} ([fX, gY]_p h) &= f(p)X_p(g(Yh)) - g(p)Y_p(f(Xh)) \\ &= f(p)(X_p g)(Y_p h) + f(p)g(p)X_p(Yh) \\ &\quad - g(p)(Y_p f)(X_p h) - g(p)f(p)Y_p(Xh) \\ &= f(p)g(p)([X,Y]_p h) + f(p)(X_p g)(Y_p h) - g(p)(Y_p f)(X_p h). \end{aligned}$$

b) Show that the Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G determines a left-invariant involutive distribution.

Remark: Part a) is not necessarily needed for part b).

Solution. Let $\iota : H \hookrightarrow G$ be a Lie subgroup and let $X_1, \dots, X_n$ be a basis of $T_e H \cong \mathfrak{h}$. We define smooth left-invariant vector fields $Y_1, \dots, Y_n$ on G via

$$(Y_i)_g = d_e L_g(d_e\iota\, X_i)$$

for every $g \in G$, $i = 1, \dots, n$. These clearly define a global basis of the left-invariant distribution $\mathcal{D} = \mathrm{span}\{Y_1, \dots, Y_n\} \subset TG$ on G.

We need to see that $\mathcal{D}$ is involutive. Observe that Y_i is L_g-related to itself for every $g \in G$ by definition. By exercise 1 also $[Y_i, Y_j]$ is L_g-related to itself such that

$$[Y_i, Y_j]_g = [Y_i, Y_j]_{L_g(e)} = d_e L_g([Y_i, Y_j]_e)$$

for every $g \in G$. Further Y_i is ι-related to X_i by definition. Therefore also $[Y_i, Y_j]$ is ι-related to $[X_i, X_j]$ such that

$$[Y_i, Y_j]_e = [Y_i, Y_j]_{\iota(e)} = d_e\iota [X_i, X_j]_e \in \mathcal{D}_e.$$

Hence,

$$[Y_i, Y_j]_g = d_e L_g([Y_i, Y_j]_e) \in d_e L_g(\mathcal{D}_e) = \mathcal{D}_g$$

by left-invariance. This shows that $\mathcal{D}$ is involutive.

Exercise 4 (Surjectivity of the Matrix Exponential)**.** Let $\mathrm{Exp} : \mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n} \to \mathrm{GL}(n,\mathbb{R})$ be the matrix exponential map given by the power series

$$\mathrm{Exp}(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Consider the Lie subgroup of upper triangular matrices $N(n) < \mathrm{GL}(n,\mathbb{R})$ with its Lie algebra $\mathfrak{n}(n) < \mathfrak{gl}(n,\mathbb{R})$ of strictly upper triangular matrices; cf. exercise sheet 4 problem 3.

Show that $\mathrm{Exp}|_{\mathfrak{n}(n)} : \mathfrak{n}(n) \to N(n)$ is surjective.

Hint: Consider the partially defined matrix logarithm $\mathrm{Log} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$\mathrm{Log}(I + A) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n}.$$

Try to give answers to the following questions and then conclude:

What is its radius of convergence r about I? Why is it a right-inverse of Exp on the ball $B_r(I)$ of radius r about I? Why is there no problem for matrices that are in $N(n)$ but not in $B_r(I)$?

In order to answer the last question prove that $A^n = 0$ for all $A \in \mathfrak{n}(n)$.

Solution. Note that

$$r = \lim_{n\to\infty} \left| \frac{(-1)^{n-1}}{n} \cdot \frac{n+1}{(-1)^n} \right| = 1$$

such that the power series $\mathrm{Log}(I + A)$ converges absolutely for every $A \in \mathbb{R}^{n\times n}$ with $\|A\| < 1$ as in the complex case.

For all complex numbers $z \in \mathbb{C}$ with $|z| < 1$ we have

$$e^{\log(1+z)} = 1 + z. \tag{1}$$

Recall that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}$$

and

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \qquad \forall z \in B_1(0) \subset \mathbb{C}.$$

Writing the composition $e^{\log(1+z)}$ as a power series we obtain

$$e^{\log(1+z)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \right)^k = \sum_{k=0}^{\infty} d_k z^k$$

for all $z \in B_1(0) \subset \mathbb{C}$ for some $d_k \in \mathbb{R}$, where one uses succesively the Cauchy product rule for power series to compute the power series representation of $\left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}\right)^k$ and then uses the fact that the series converges absolutely for $|z| < 1$ to reorder it and to obtain the coefficients for each z^k.

Comparing coefficients in (1) then yields that $d_0 = d_1 = 1$ and $d_k = 0$ for all $k > 1$.

Let us now write $\mathrm{Exp}(\mathrm{Log}(I + A))$ as well as a power series

$$\mathrm{Exp}(\mathrm{Log}(I + A)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n} \right)^k = \sum_{k=0}^{\infty} d_k A^k$$

for all $z \in B_1(0) \subset \mathbb{C}$, where one uses succesively the Cauchy product rule for power series to compute the power series representation of $\left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}\right)^k$ and then uses the fact that the series converges absolutely for $|z| < 1$ to reorder it and to obtain the coefficients for each z^k as above.

Observe that the coefficients $d_k \in \mathbb{R}$ are the same as in the complex case! This is due to the fact that they arise from the same computation with power series (Cauchy product rule and reordering accordingly). Hence, $d_0 = d_1 = 1$ and $d_k = 0$ for all $k > 1$ such that

$$\mathrm{Exp}(\mathrm{Log}(I + A)) = I + A$$

for every $A \in \mathbb{R}^{n\times n}$ with $\|A\| < 1$. [1]

Finally, observe that every $X \in N(n)$ can be written as $X = I + A$ where $A \in \mathfrak{n}(n)$. Furthermore, since A is strictly upper triangular it maps

$$A|_{V_i} : V_i \to V_{i-1}$$

[1] The reasoning applied here can be generalized. In fact, there are theorems that relate identities of complex power series to identities of power series in Banach algebras; see e.g. Königsberger: „Analysis 2”, ch. 1.6 and Königsberger: „Analysis 2”, Exercise 18, p. 44

where $V_i = \mathrm{span}\{e_1, \dots, e_i\}$, $V_0 = \{0\}$ for every $i = 1, \dots, n$. In particular,

$$A^n : \mathbb{R}^n = V_n \to V_{n-1} \to \cdots \to V_0 = \{0\}$$

such that $A^n = 0$.

That means that for every $A \in \mathfrak{n}(n)$ the power series $\mathrm{Log}(I + A)$ is actually a polynomial in A taking values in $\mathfrak{n}(n)$:

$$\mathrm{Log}(I + A) = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{A^k}{k} \in \mathfrak{n}(n).$$

Because $\mathrm{Log}(I + A)$ is again in $\mathfrak{n}(n)$ also $\mathrm{Exp}(\mathrm{Log}(I + A))$ becomes a polynomial p in A:

$$\mathrm{Exp}(\mathrm{Log}(I + A)) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left(\sum_{l=1}^{n-1} (-1)^{l-1} \frac{A^l}{l} \right)^k}_{=0, \text{ if } k \geq n} = \sum_{k=0}^{n-1} \frac{1}{k!} \left(\sum_{l=1}^{n-1} (-1)^{l-1} \frac{A^l}{l} \right)^k =: p(A)$$

Now observe that $\|tA\| < 1$ for all $t \in I_A := (-\|A\|^{-1}, \|A\|^{-1}) \subset \mathbb{R}$. Hence,

$$p(tA) = \mathrm{Exp}(\mathrm{Log}(I + tA)) = I + tA$$

for all $t \in I_A$. The left-hand-side and the right-hand-side are both polynomials in t which coincide on an open subset of $\mathbb{R}$. Thus they have to coincide everywhere; in particular

$$\mathrm{Exp}(\mathrm{Log}(I + A)) = I + A$$

for $t = 1$. This shows that $\mathrm{Log}|_{N(n)}$ is a well-defined right-inverse of $\mathrm{Exp}|_{\mathfrak{n}(n)}$.

Exercise 5. Continuous one-parameter subgroups Let G be a Lie group and $\varphi : \mathbb{R} \to G$ a continuous group homomorphism. Prove that φ is smooth.

Hint: Use the exponential map.

Solution. We claim that there is $Z \in \mathfrak{g}$ such that

$$\varphi(t) = \exp(tZ) \quad (t \in \mathbb{R}).$$

To this end let $V \subseteq \mathfrak{g}$ an open ball around 0 such that $U := \exp(V)$ is open in G and $\exp\big|_V$ is a diffeomorphism onto its image. Let $V' := \frac{1}{2}V$ and $U' := \exp(V')$. As φ is continuous, there is $\varepsilon > 0$ such that $\varphi(-\varepsilon, \varepsilon) \subseteq U'$. Let $0 < |t_0| < \varepsilon$, $n \in \mathbb{N}$ arbitrary and choose $Y, X \in V'$ such that

$$\varphi(t_0) = \exp(Y), \quad \varphi\left(\frac{t_0}{n}\right) = \exp(X).$$

We claim that $nX = Y$. Assume that we know for $1 \leq k < n$ that $kX \in V'$, then $(k+1)X \in V$ and

$$\exp\big((k+1)X\big) = \exp(X)^{k+1} = \varphi\left(\frac{t_0}{n}\right)^{k+1} = \varphi\left(\frac{k+1}{n} t_0\right) \in U'$$

as $\left|\frac{k+1}{n} t_0\right| \leq |t_0| < \varepsilon$. This implies that there is some $X_{k+1} \in V'$ such that

$$\exp\big((k+1)X\big) = \exp(X_{k+1}).$$

As $(k+1)X \in V \supset V'$ and $\exp\big|_V$ is injective, it follows $X_{k+1} = (k+1)X$ and hence $(k+1)X \in V'$. In particular $nX \in V'$. Now

$$\exp(nX) = \exp(X)^n = \varphi\left(\frac{t_0}{n}\right)^n = \varphi(t_0) = \exp(Y)$$

and once more injectivity of $\exp\big|_V$ implies $nX = Y$. Now let $\frac{p}{q} \in \mathbb{Q}$, then

$$\varphi\left(\frac{p}{q}t_0\right) = \varphi\left(\frac{t_0}{q}\right)^p = \exp\left(\frac{1}{q}Y\right)^p = \exp\left(\frac{p}{q}Y\right).$$

Continuity yields

$$\varphi(r) = \exp(rZ) \quad (r \in \mathbb{R})$$

with $Z := \frac{1}{t_0}Y$. This proves smoothness of φ.