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Solutions of Exercise Sheet 5

Exercise 1 (Abstract Subgroups as Lie Subgroups)**.** Let H be an abstract subgroup of a Lie group G and let $\mathfrak{h}$ be a subspace of the Lie algebra $\mathfrak{g}$ of G. Further let $U \subseteq \mathfrak{g}$ be an open neighborhood of $0 \in \mathfrak{g}$ and let $V \subseteq G$ be an open neighborhood of $e \in G$ such that the exponential map $\exp : U \to V$ is a diffeomorphism satisfying $\exp(U \cap \mathfrak{h}) = V \cap H$. Show that the following statements hold:

a) H is a Lie subgroup of G with the induced relative topology;

b) $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;

c) $\mathfrak{h}$ is the Lie algebra of H.

Remark: This is precisely the lemma we saw in class but did not prove.

Solution. We will first show that H is an embedded submanifold of G. For that it is enough to check that there are slice charts about every point $h \in H$. For $h = e$ choose any linear isomorphism $E : \mathfrak{g} \to \mathbb{R}^m$ that sends $\mathfrak{h}$ to $\mathbb{R}^k$ where $\dim G = \dim \mathfrak{g} = m$ and $\dim \mathfrak{h} = k$. The composite map

$$\varphi = E \circ \exp^{-1} : \exp U = V \longrightarrow \mathbb{R}^m$$

is then a smooth chart for G, and

$$\varphi((\exp(U) \cap H) = E(U \cap \mathfrak{h})$$

is the slice obtained by setting the last $m-k$ coordinates equal to zero. Moreover, if $h \in H$ is arbitrary, the left translation map L_h is a diffeomorphism from $\exp(U)$ to a neighborhood of h. Since H is a subgroup, $L_h(H) = H$, and so

$$L_h((\exp U) \cap H) = L_h(\exp U) \cap H,$$

and $\varphi \circ L_h^{-1}$ is easily seen to be a slice chart for H in a neighborhood of h. Thus H is an embedded submanifold of G.

We will now make use of the following Lemma:

Lemma: Let G be a Lie group, and suppose $H \subseteq G$ is a subgroup that is also an embedded submanifold. Then H is a Lie subgroup.

Proof: We need only check that multiplication $m : H \times H \to H$ and inversion $i : H \to H$ are smooth maps. Because multiplication is a smooth map from $G \times G$ to G its restriction is clearly smooth from $H \times H$ to G. Because H is a subgroup, multiplication takes $H \times H$ to H. Using local slice charts for H in G it follows easily that $m : H \times H \to H$ is smooth. The same argument works for inversion. □

This proves a). We will prove b) and c) in one go:

Denote by $\iota : H \to G$ the embedding from H into G and let $\mathfrak{b} \subseteq \mathfrak{g}$ be a complementary subspace of $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$. This yields the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Lie}(H) & \xrightarrow{d_e\iota} & \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b} \\ \big\downarrow \exp & & \big\downarrow \exp \\ H & \xrightarrow{\quad\iota\quad} & G \end{array}$$

By construction of the slice charts of H it is immediate that $d_e\iota$ is an isomorphism of vector spaces from $\mathrm{Lie}(H)$ to $\mathfrak{h}$. Furthermore, ι is a Lie group homomorphism whence its differential $d_e\iota$ induces a Lie algebra homomorphism. Therefore $d_e\iota$ is a Lie algebra isomorphism from $\mathrm{Lie}(H)$ to $\mathfrak{h}$. Under the identification $H \cong \iota(H) \leq G$ we get $\mathrm{Lie}(H) \cong \mathfrak{h}$. This proves b) and c).

Exercise 2 (Quotients of Lie groups)**.** Let G be a Lie group and let $K \leq G$ be a closed normal subgroup.

Show that G/K can be equipped with a Lie group structure such that the quotient map $\pi\colon G \to G/K$ is a surjective Lie group homomorphism with kernel K.

Solution. From the lecture we know that there exists a suitable neighborhood $U \subset \mathfrak{g}$ of the origin such that $\exp|_U : U \to \exp(U)$ is a diffeomorphism. Denote by $\mathfrak{k} = \mathrm{Lie}(K)$ the Lie algebra associated to K. Choose any complement $\mathfrak{l}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$ as vector spaces. Define

$$V := U \cap \mathfrak{l}.$$

Since $V \cap \mathfrak{k} = \{0\}$ it is immediate to verify that $\pi \circ \exp|_V : V \to G/K$ is a homeomorphism onto the image. This gives us a local chart around the point $K \in G/K$. We can get an atlas by suitably translating this chart by the natural action of G on G/K. This gives us back an atlas such that each change of coordinate charts is smooth (since the multiplication in G is smooth).

Note that multiplication and inversion are defined on G/K by passing to the quotient, i.e. the following diagrams commute:

$$\begin{array}{ccc} G \times G & \xrightarrow{\;m\;} & G \\ \big\downarrow{\scriptstyle \pi\times\pi} & & \big\downarrow{\scriptstyle \pi} \\ G/K \times G/K & \dashrightarrow & G/K \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\;i\;} & G \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ G/K & \dashrightarrow & G/K \end{array}$$

By definition, the quotient map $\pi\colon G \to G/K$ is a smooth submersion with respect to this smooth structure. Thus, it follows from the constant rank theorem that multiplication and inversion are smooth, and G/K is a Lie group. Moreover, it is clear from the construction that K is the kernel of π.

For more details see Theorem 21.26 in *John M. Lee, "Intorduction to Smooth Manifolds", Springer (2013)*

Exercise 3 ($\mathrm{Z}(G) = \mathrm{Ker}(\mathrm{Ad})$)**.** Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Use the fundamental relation that

$$g \exp(tX) g^{-1} = \exp(t\,\mathrm{Ad}_g(X))$$

for all $g \in G, t \in \mathbb{R}$ and $X \in \mathfrak{g}$ to prove the following.

(1) If G is connected, then the center $\mathrm{Z}(G)$ of G equals the kernel of the adjoint representation.

Solution. If $g \in \mathrm{Z}(G)$, then we have for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$ that

$$\exp(tX) = g \exp(tX) g^{-1} = \exp(t\,\mathrm{Ad}_g(X))$$

and while exp may not be injective on all of $\mathfrak{g}$, it is injective on an open neighborhood of 0, in particular there is a vector-space basis of $\mathfrak{g}$ contained in the open neighborhood of 0 such that $X = \mathrm{Ad}_g(X)$ for all elements of the basis. By linear extension, we then have that $\mathrm{Ad}_g = \mathrm{Id}$, so $g \in \mathrm{Ker}(\mathrm{Ad})$.

If on the other hand we start with $g \in \mathrm{Ker}(\mathrm{Ad})$, we apply the same formula to see that g commutes with all elements in an open neighborhood of $e \in G$ (contained in $\exp(\mathfrak{g}) \subseteq G$). Since G is connected, every element $h \in G$ is of the form $h = h_1 h_2 \cdots h_n$ for h_i in the neighborhood, and since g commutes with h_i individually, it commutes with h, so $g \in \mathrm{Z}(G)$.

(2) If G is connected, $\mathrm{Z}(G)$ is a closed subgroup and

$$\mathrm{Lie}(\mathrm{Z}(G)) = \mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \colon \forall Y \in \mathfrak{g}, [X, Y] = 0\}.$$

Solution. Note that $\mathrm{Z}(G) = \{g \in G \colon \forall h \in G, ghg^{-1}h^{-1} = e\}$, so we can write

$$\mathrm{Z}(G) = \bigcap_{h \in G} f_h^{-1}(e) \quad \text{for} \quad f_h(g) = ghg^{-1}h^{-1}$$

as a closed subgroup. By Proposition 3.18 we have

$$\mathrm{Lie}(\mathrm{Z}(G)) = \mathrm{Lie}(\mathrm{Ker}(\mathrm{Ad})) = \mathrm{Ker}(\mathrm{D}\,\mathrm{Ad}) = \mathrm{Ker}(\mathrm{ad}) = \mathfrak{z}(\mathfrak{g})$$

since $\mathrm{ad}_X(Y) = [X, Y]$.

Exercise 4 (The adjoint representation ad)**.** Let V be a vector space over a field k.

a) Show that the vector space of endomorphisms

$$\mathfrak{gl}(V) \coloneqq \{A \colon V \to V \text{ linear}\}$$

is a Lie algebra with the Lie bracket given by the commutator

$$[A, B] \coloneqq AB - BA$$

for all $A, B \in \mathfrak{gl}(V)$.

Solution. One immediately verifies that $\mathfrak{gl}(V)$ is an algebra with respect to the Lie bracket. What is left to check is that the commutator satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{gl}(V)$.

We leave this computation to the reader.

b) Let $\mathfrak{g}$ be a Lie algebra over k. The *adjoint representation*

$$\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is defined as $\mathrm{ad}(X)(Y) \coloneqq [X, Y]$ for all $X, Y \in \mathfrak{g}$. Show that ad is a Lie algebra homomorphism.

Solution. It is easy to check that $\mathrm{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is linear. Thus, we only need to check that it preserves the Lie bracket.

We compute

$$\begin{aligned}
[\mathrm{ad}(X), \mathrm{ad}(Y)](Z) &= (\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X))(Z) \\
&= [X, [Y, Z]] - [Y, [X, Z]] \\
&= [X, [Y, Z]] + [Y, [Z, X]] \\
(\text{Jacobi identity}) &= -[Z, [X, Y]] \\
&= [[X, Y], Z] = \mathrm{ad}([X, Y])(Z)
\end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$.

Exercise 5. Adjoint of nilpotent elements Let $\mathfrak{g} \leq \mathfrak{gl}_n(\mathbb{C})$ be a Lie subalgebra.

Show that, if $X \in \mathfrak{g}$ is nilpotent then $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.

Solution. This will follow from the following formula

$$\mathrm{ad}(X)^n(Y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{n-k} Y X^k \tag{1}$$

for every $X, Y \in \mathfrak{g}$, $n \geq 0$.

Indeed, $X \in \mathfrak{g}$ is nilpotent if and only if $X^m = 0$ for some $m \in \mathbb{N}$. Then, by the above formula (1), $\mathrm{ad}(X)^{2m}(Y) = 0$ for every $Y \in \mathfrak{g}$.

We will prove (1) by induction on n. For $n = 0$ there is nothing to show. So, let us assume that (1) holds for n and we want to prove it for $n+1$. This is a direct computation:

$$\begin{aligned}
\mathrm{ad}(X)^{n+1}(Y) &= \mathrm{ad}(X)\left(\mathrm{ad}(X)^n(Y)\right) \\
&= X \cdot \mathrm{ad}(X)^n(Y) - \mathrm{ad}(X)^n(Y) \cdot X \\
&= X \cdot \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{n-k} Y X^k\right) - \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{n-k} Y X^k\right) \cdot X \\
&= X^{n+1} Y + \sum_{k=1}^{n} (-1)^k \binom{n}{k} X^{n-k+1} Y X^k - \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} X^{n-k} Y X^{k+1} + (-1)^{n+1} Y X^{n+1} \\
&= X^{n+1} Y + \sum_{k=1}^{n} (-1)^k \left(\binom{n}{k} + \binom{n}{k-1}\right) X^{n-k+1} Y X^k + (-1)^{n+1} Y X^{n+1} \\
&= X^{n+1} Y + \sum_{k=1}^{n} (-1)^k \binom{n+1}{k} X^{n-k+1} Y X^k + (-1)^{n+1} Y X^{n+1} \\
&= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} X^{n+1-k} Y X^k
\end{aligned}$$