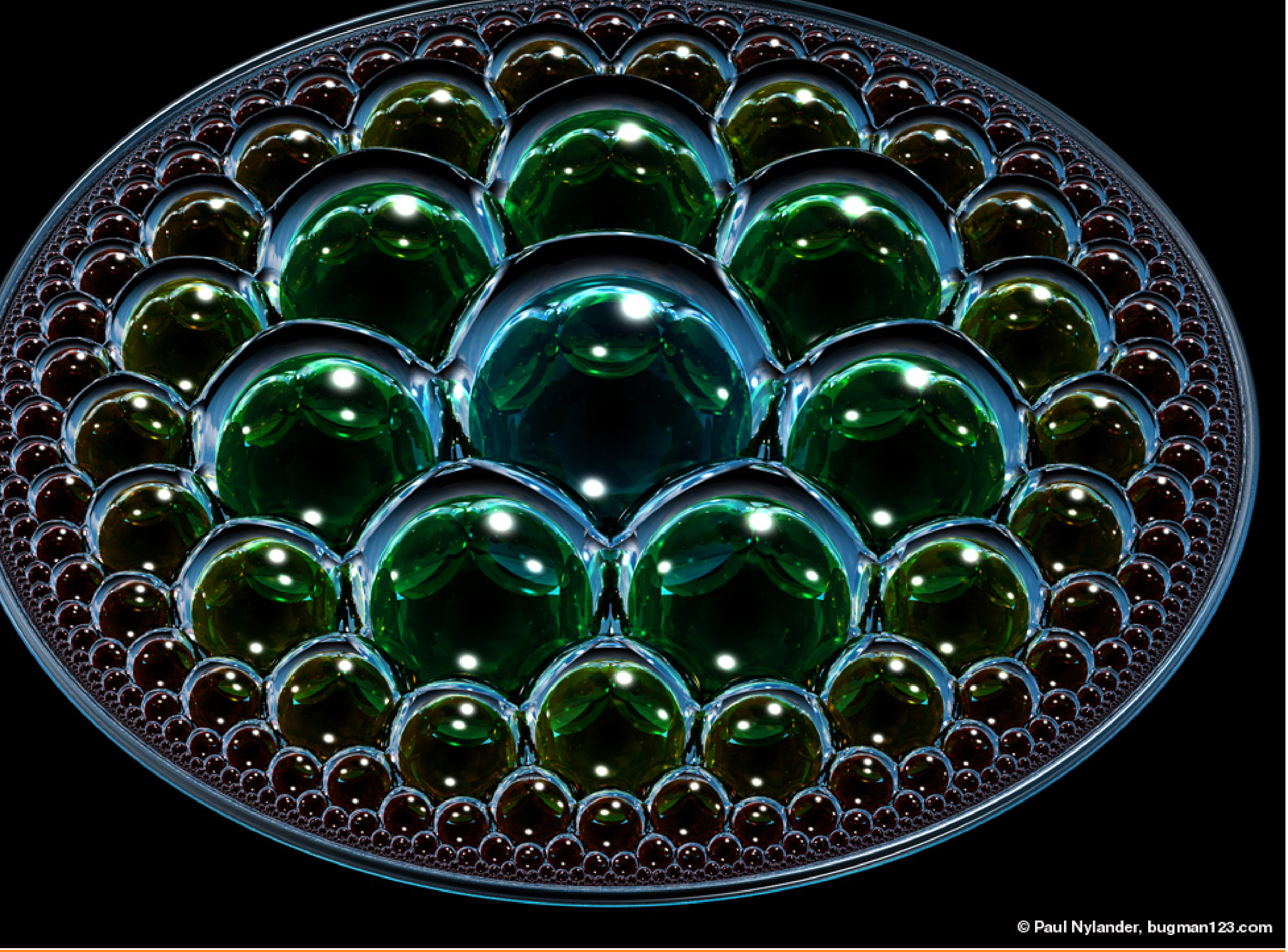

From Topological Groups to Lie Groups

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Chapter 1 Introduction

Lie groups are named after Sophus Lie, a Norwegian mathematician of the second half of the nineteenth century who developed the theory of continuous transformation groups. His original idea was to develop a theory of symmetries of differential equations parallel to the theory developed by Galois for algebraic equations, with Lie groups being the continous analogue of permutation groups in Galois theory. This point of view did not fulfill Lie's expectations and went in unexpected directions (see for example the theory of differential fields, D-modules, etc), but Lie groups came to be an indispensable tool in many branches of mathematics as well as in theoretical physics.

The definition of a Lie group is simple, in that it is a differentiable manifold that is also a group, such that the group operations are compatible with the manifold structure. One can hence study Lie groups from a geometrical point of view or from an algebraic point of view. The starting point of the algebraic point of view is the existence of an algebraic object, namely the Lie algebra of the Lie group, that turns out to have the geometric interpretation as the tangent space at the identity of the Lie group. There is also a somewhat different approach, much more elementary, but also more restrictive, in which one considers only linear Lie groups, that is (closed) subgroups of $\mathrm{GL}(n, \mathbb{R})$ and one develops the whole theory via elementary methods. While this is appropriate for example in case one is teaching a course to students with limited mathematical background, there is the problem that, although any Lie group can be locally realized as a linear Lie group, there are important Lie groups that are not (globally) linear. Our approach will be the more algebraic one. In fact, as such we will start by considering Lie groups just as topological groups, that is topological spaces that are also groups, such that the group operations are compatible with the topological structure. We will see how far one can go for topological groups, and we will see that there are some miraculous facts that arise from the interplay of these two structures.

Chapter 2 Topological Groups

1. Check Exercise numbering

2.1 Definitions and Examples

Definition 2.1. Topological group

A topological group G *is a group endowed with a topology with respect to which the group operations*

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are continuous, where $G \times G$ is endowed with the product topology. ♣

Remark The following "regularity" properties follow simply from the definition:

- The inversion $g \mapsto g^{-1}$ is a continuous bijection. Since its inverse $g^{-1} \mapsto (g^{-1})^{-1}$ is also continuous, then it is a homeomorphism.
- The left translation

$$\begin{aligned} L_g \colon G &\to G \\ x &\mapsto gx \end{aligned}$$

  and the right translation

$$\begin{aligned} R_g \colon G &\to G \\ x &\mapsto xg \end{aligned}$$

  are continuous and bijective. Since $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$ are also continuous, L_g and R_g are homeomorphisms. If $U \ni e$ is a neighborhood of the identity (that is a set in G containing e and an open set $U_e \ni e$), then $L_g U$ is a neighborhood of g homeomorphic to

U. Hence topological groups "look everywhere the same".

- If G_1, G_2 are topological groups and $\rho\colon G_1 \to G_2$ is a homomorphism, then ρ is continuous if and only if it is continuous at one point.

Remark. Before we proceed to give concrete examples of topological groups, we remark that there are simple operations that preserve the class of topological groups.

- Any subgroup of a topological group is a topological group (see also Proposition 2.1.3.).
- Products of topological groups are topological groups with the product topology.
- Quotients of topological groups are also topological groups with the quotient topology.
- The semidirect product of topological groups is a topological group. We recall in fact that if H, N are topological groups and $\eta\colon H \to \mathrm{Aut}(N)$ is a homomorphism such that
$$\begin{aligned} H \times N &\to N \\ (h, n) &\mapsto \eta(h)n \end{aligned}$$
is continuous, the semidirect product $H \ltimes_\eta N$ is the setwise Cartesian product $H \times N$ with the product
$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 \eta(h_1) n_2)$$
for all $h_1, h_2 \in H$ and $n_1, n_2 \in N$, and it is a topological group with the product topology. Notice that there are other characterizations of a semidirect product. We recall these here since we will be using it in the sequel.

Lemma 2.1

Let G be a topological group, $H < G$ a closed subgroup and $N \trianglelefteq G$ a closed normal subgroup. The following are equivalent:

1. *There exists a homomorphism $\eta\colon H \to \mathrm{Aut}(N)$ such that $G = H \ltimes_\eta N$;*
2. *G is a group extension of N by H, that is there exists a short exact sequence*
$$\{e\} \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow \{e\}\,.$$
that splits, that is the composition $p \circ i\colon H \to G/N$ of the embedding $i\colon H \hookrightarrow G$ and the natural projection $p\colon G \to G/N$ is an isomorphism of topological groups.

♡

Example 2.1 Any group with the discrete topology is a topological group. In this case any subset is open and any map to any other topological group is continuous.

Example 2.2 The vector space $(\mathbb{R}^n, +)$ with the componentwise addition is a commutative

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topological group in the Euclidean topology.

Example 2.3 The non-zero real numbers and the non-zero complex numbers, $(\mathbb{R}^*, \cdot)$ and $(\mathbb{C}^*, \cdot)$, are commutative topological groups with the topology induced by the Euclidean topology.

Example 2.4 Let us denote by $\mathbb{R}^{n\times n}$ the vector space of $n \times n$ matrices with real coefficients and let us define

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\} .$$

Then $\mathrm{GL}(n, \mathbb{R})$ is an open set in $\mathbb{R}^{n\times n}$ and it inherits from $\mathbb{R}^{n\times n}$ the Euclidean topology. With this topology $\mathrm{GL}(n, \mathbb{R})$ is a topological group. In fact the topology on $\mathbb{R}^{n\times n}$, and hence on $\mathrm{GL}(n, \mathbb{R})$ is such that if $(A_k)_{k\in\mathbb{N}} \subset \mathrm{GL}(n, \mathbb{R})$ is a sequence, then

$$A_k \to A \text{ if and only if } (A_k)_{ij} \to A_{ij}$$

for all $1 \leq i, j \leq n$. Since if $A, B \in \mathrm{GL}(n, \mathbb{R})$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} ,$$

this means that the multiplication is continuous. Since

$$(A^{-1})_{ij} = \frac{\det M_{ji}}{\det A} ,$$

where M_{ji} is the (j, i)-minor matrix obtained by removing the i-th row and the j-th column and by multiplying by $(-1)^{i+j}$, then the inversion is also continuous.

Example 2.5 In the Example 2.4 we used that $\mathbb{R}$ is a *topological field*, that is the sum, the multiplication and the inversion are continuous, and as a consequence, the topology on $\mathbb{R}^{n\times n}$ induces a topology on $\mathrm{GL}(n, \mathbb{R})$. Likewise, if $\mathbb{F}$ is any topological field, then $\mathrm{GL}(n, \mathbb{F})$ is a topological group. Examples of topological fields are $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ and finite fields. Here $\mathbb{Q}_p$ is the field of p-adic integers, which can be defined as the field of fractions of the ring of p-adic integers $\mathbb{Z}_p$ defined in Example 2.10.

Example 2.6 Let X be a compact Hausdorff space. Then

$$\mathrm{Homeo}(X) := \{f : X \to X : f \text{ is a homeomorphism}\}$$

is a topological group with the compact-open topology (see Definition A.5).

If X is only locally compact but not compact, then $\mathrm{Homeo}(X)$ need not be a topological group. If however X is locally compact but also locally connected, then $\mathrm{Homeo}(X)$ is a topological group. This includes for example all manifolds. Likewise, if X is a proper metric space (that is a metric space in which closed balls of finite radius are compact), then $\mathrm{Homeo}\, X$ is a topological group.

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Example 2.7 Let (X, d) be a compact metric space and let

$$\mathrm{Iso}(X) := \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X\}.$$

Then $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is a (closed) subgroup and hence a topological group (Exercise 1.).

Example 2.8 We showed in Example 2.4 that $\mathrm{GL}(n, \mathbb{R})$ is a topological group when it inherits the Euclidean topology as a subspace of $\mathbb{R}^{n\times n}$. We show now that $\mathrm{GL}(n, \mathbb{R})$ is a topological group also with respect to the compact-open topology (and in fact the two topologies coincide, see Exercise 2.). In fact, since $\mathrm{GL}(n, \mathbb{R}) < \mathrm{Homeo}(\mathbb{R}^n)$ and $\mathbb{R}^n$ is a proper metric space, then the compact-open topology on $\mathrm{GL}(n, \mathbb{R}) < \mathrm{Homeo}(\mathbb{R}^n)$ is the topology of the uniform convergence on compact sets. If $(A_k)_{k\in\mathbb{N}} \subset \mathrm{GL}(n, \mathbb{R})$, and $A_k \to A$ uniformly on compact set, then A is linear, so that $\mathrm{GL}(n, \mathbb{R})$ is a (closed) subgroup of $\mathrm{Homeo}(\mathbb{R}^n)$ and is hence a topological group. Notice that for the limit of a sequence of linear functions to be linear, it is actually enough that the sequence converges pointwise.

Example 2.9 Let M be a smooth manifold. Then

$$\mathrm{Diff}^r(M) := \{f \in \mathrm{Homeo}(M) : f, f^{-1} \text{ are continuous and differentiable } r \text{ times}\}$$

is a subgroup of $\mathrm{Homeo}(M)$, hence a topological group, which however is not closed in the compact-open topology. We can consider however the C^r-topology, that is the topology according to which $(f_n)_{n\in\mathbb{N}} \xrightarrow{C^r} f$ if in any local chart $\psi\colon U \to \mathbb{R}^n$, $U \subset M$, the sequence $(f_n \circ \psi^{-1})_{n\geq 1}$ and all its partial derivatives up to order r converge uniformly on compact sets to the corresponding derivatives of $f \circ \psi^{-1}$. With this topology $\mathrm{Diff}^r(M)$ is a topological group that is complete in a natural sense.

Example 2.10 Let Λ be a partially ordered set and let $(G_\lambda)_{\lambda\in\Lambda}$ be a family of groups such that for every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \leq \lambda_2$ there exists a homomorphism

$$G_{\lambda_2} \xrightarrow{\rho_{\lambda_2,\lambda_1}} G_{\lambda_1}$$

satisfying the following properties:

1. for any $\lambda \in \Lambda$, $\rho_{\lambda,\lambda} = \mathrm{Id}|_{G_\lambda}$;
2. for any $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3 \leq \lambda_2$;
3. $\rho_{\lambda_3,\lambda_1} = \rho_{\lambda_2,\lambda_1} \circ \rho_{\lambda_3,\lambda_2}$ for all $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

Then the *inverse limit* G of the *projective system* $((G_\lambda)_{\lambda\in\Lambda}, \rho_{\lambda_2,\lambda_1})$ is defined as the unique smallest topological group G such that for all $\lambda \in \Lambda$ there exists a continuous homomorphism $\rho_\lambda\colon G \to G_\lambda$

with the property that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho_{\lambda_2}} & G_{\lambda_2} \\ & {\scriptstyle\rho_{\lambda_1}}\searrow \quad \swarrow{\scriptstyle\rho_{\lambda_2,\lambda_1}} & \\ & G_{\lambda_1} & \end{array} \tag{2.1}$$

commutes, $\rho_{\lambda_1} = \rho_{\lambda_2,\lambda_1} \circ \rho_{\lambda_2}$. One can verify that G can be identified with

$$\varprojlim G_\lambda := \left\{ (x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} G_\lambda \,:\, \rho_{\lambda_2,\lambda_1}(x_{\lambda_2}) = x_{\lambda_1} \right\} .$$

Points in $\varprojlim G_\lambda$ are said to be *compatible*. If the $(G_\lambda)_{\lambda\in\Lambda}$ are topological groups, so is $\prod_{\lambda\in\Lambda} G_\lambda$ with the product topology and, since $\varprojlim G_\lambda$ is a (closed) subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$, it is a topological group as well with the induced topology.

Of course if the $(G_\lambda)_{\lambda\in\Lambda}$ are compact then by Tychonoff theorem also $\varprojlim G_\lambda$ is compact. Moreover, if the $(G_\lambda)_{\lambda\in\Lambda}$ are discrete, then $\varprojlim G_\lambda$ is totally disconnected, that is the connected sets are the points. In fact, let $C \subset G$ be a connected set. Since $\rho_\lambda\colon G \to G_\lambda$ is continuous, then $\rho_\lambda(C)$ is connected and hence a point, say $x_\lambda \in G_\lambda$. By the commutativity of the diagram (2.1) the sequence $(x_\lambda)_{\lambda\in\Lambda}$ must be compatible and unique, so that C is the singleton $\{(x_\lambda)_{\lambda\in\Lambda}\}$.

If the groups in the projective system $(G_\lambda)_{\lambda\in\Lambda}$ are finite, the resulting inverse limit is called *profinite*. It follows from the previous observation that profinite groups are compact and totally disconnected. An important example is the group of *p-adic integers* $\mathbb{Z}_p$, which is a profinite group under addition. In fact $\mathbb{Z}_p$ is the inverse limit of the projective system

$$((\mathbb{Z}/p^n\mathbb{Z}), (\rho_{n,m}\colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z})_{n\geq m}) ,$$

where $\rho_{n,m}$ is the natural reduction $\mod p^m$ homomorphism. One can check that the topology on $\mathbb{Z}_p$ is the same as the topology arising from the p-adic valuation on $\mathbb{Z}_p$ and with this topology $\mathbb{Z}_p$ is a topological ring. By the characteristic property of $\mathbb{Z}_p$ there are maps $\rho_n\colon \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ which hare continuous ring homomorphisms. The kernel of ρ_n is the ideal $p^n\mathbb{Z}_p$ which is open since $\mathbb{Z}/p^n\mathbb{Z}$ is discrete. Since $\bigcap_{n\geq 1} p^n\mathbb{Z}_p = \{0\})$, the sequence $\{p^n\mathbb{Z}_p \,:\, n \geq p\}$ is a fundamental system of neighborhoods of 0. One shows then:

1. $x \in \mathbb{Z}_p$ is invertible if and only if $x \notin p\mathbb{Z}_p$;
2. if $U = \mathbb{Z}_p^\times$ is the group of invertible elements, then every $x \in \mathbb{Z}_p \setminus \{0\}$ can be written uniquely as $x = p^n u$, with $n \geq 0$ and $u \in U$.

With this at hand, one shows that $\mathbb{Z}_p$ is an integral domain; its field of fractions is the field $\mathbb{Q}_P$ of

p-adic numbers and equals $\mathbb{Z}_p\left[\frac{1}{p}\right]$. It is a locally compact non-discrete Hausdorff field. In fact any such field of characteristic zero is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$. For more details see [9].

Example 2.11 We consider now three subgroups of $\mathrm{GL}(n,\mathbb{R})$ that will turn out to play an extremely important role.

1. Let
$$A_{\det} := \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) : \lambda_i \neq 0, \text{ for } 1 \leq i \leq n \right\}.$$
Then $A_{\det}$ is an Abelian topological group as it is homomorphic and homeomorphic to $(\mathbb{R}^*)^n$.
2. Let
$$N := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathrm{GL}(n,\mathbb{R}) \right\}.$$
be the group of upper triangular matrices with all 1s on the diagonal. Then N is a (closed) subgroup of $\mathrm{GL}(n,\mathbb{R})$ and is hence a topological group. However, in this case N is homeomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$ as a topological space, but not as a group, as for example N is not Abelian, unless $n \leq 2$.
3. Let
$$\begin{aligned} K := \mathrm{O}(\mathbb{R}^n, \langle\cdot\,,\cdot\rangle) &= \{X \in \mathrm{GL}(n,\mathbb{R}) : \langle Xv, Xw\rangle = \langle v, w\rangle \text{ for all } v, w \in \mathbb{R}^n\} \\ &= \{X \in \mathrm{GL}(n,\mathbb{R}) : \|Xv\| = \|v\| \text{ for all } v \in \mathbb{R}^n\} \\ &= \left\{X \in \mathrm{GL}(n,\mathbb{R}) : {}^tXX = \mathrm{Id}_n\right\} \end{aligned}$$
be the orthogonal group of the usual Euclidean inner product $\langle\cdot\,,\cdot\rangle$ or of the usual Euclidean norm $\|\cdot\|$ on $\mathbb{R}^n$. This is a topological group as it is a (closed) subgroup of $\mathrm{GL}(n,\mathbb{R})$. The standard notation for this group is
$$\mathrm{O}(n,\mathbb{R}) := \mathrm{O}(\mathbb{R}^n, \langle\cdot\,,\cdot\rangle).$$

Example 2.12 We may also consider inner products on a vector space with respect to which vectors might have negative length. Let V be a real vector space and let $B : V \times V \to \mathbb{R}$ be a non-degenerate symmetric bilinear form on V, that is:

1. (Non-degeneracy) Given $x \in V$ there exists $y \in V$ such that $B(x, y) \neq 0$;

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2. (Symmetry) $B(v,w) = B(w,v)$ for all $v, w \in V$, and
3. (Bilinearity) $B(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 B(v_1, w) + \alpha_2 B(v_2, w)$ for all $v_1, v_1, w \in V$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$.

Incidentally, given such a non-degenerate symmetric bilinear form is equivalent to choosing a self-adjoint isomorphism $\lambda : V \to V^*$ of V with its dual V^* given by $B(v,w) = \lambda(v)w$. If Q is the quadratic form associated to B, $Q(v) := B(v,v)$, the *orthogonal group of* (V,B) or of (V,Q) is defined as

$$\begin{aligned} \mathrm{O}(V,B) &= \{A \in \mathrm{GL}(V) : \; B(Av, Aw) = B(v,w), \text{ for all } v, w \in V\} \\ &= \{A \in \mathrm{GL}(V) : \; Q(Av) = Q(v), \text{ for all } v \in V\} \, . \end{aligned}$$

This is a topological group as it is a (closed) subgroup of $\mathrm{GL}(V)$.

Recall that one can always choose a basis of V so that B can be written as

$$B_p(v,w) = -\sum_{j=1}^{p} v_j w_j + \sum_{j=p+1}^{n} v_j w_j \tag{2.2}$$

for some fixed p. Then B is positive definite if and only if $p = 0$. If $V = \mathbb{R}^n$ and B_p is as in (2.2), then it is customary to use the notation

$$\mathrm{O}(p,q) := \mathrm{O}(V, B_p) \, .$$

Notice that in the above discussion it is essential that V is a real vector space, since instead over the complex numbers all $\mathrm{O}(V,B)$ are isomorphic once the dimension of V is fixed. In fact we can perform a change of basis

$$(e_1, \dots, e_p, e_{p+1}, \dots, e_n) \mapsto (\imath e_1, \dots, \imath e_p, e_{p+1}, \dots, e_n)$$

so that in the new basis the bilinear form reads

$$B(v,w) = \sum_{j=1}^{n} v_j w_j \, . \tag{2.3}$$

The *orthogonal group of the symmetric bilinear form in* (2.3) is denoted by

$$\mathrm{O}(n, \mathbb{C}) = \mathrm{O}(V, B) \, ,$$

where now V is a complex n-dimensional vector space.

Example 2.13 Let V be a complex vector space and $h \colon V \times V \to \mathbb{C}$ a Hermitian inner product, that is a positive definite Hermitian complex valued form that is linear in the first variable and

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antilinear in the second. The *unitary group* $U(V,h)$ is defined as

$$\begin{aligned}\mathrm{U}(V,h) :=& \{X \in \mathrm{GL}(V) :\ h(Xv,Xw) = h(v,w) \text{ for all } v,w \in V\}\\ =& \{X \in \mathrm{GL}(V) :\ X^* = X^{-1}\},\end{aligned}$$

where X^* denotes the adjoint with respect to h. Notice that if $X \in \mathrm{U}(V,h)$, then $|\det X| = 1$. If $h\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is the standard Hermitian inner product

$$h(x,y) := \sum_{j=1}^{n} x_j \overline{y}_j\,,$$

then we use the notation

$$\mathrm{U}(n) := \mathrm{U}(\mathbb{C}^n, h)\,.$$

Example 2.14 Let k be a topological field. The *special linear group* defined as

$$\mathrm{SL}(n,k) := \{X \in \mathrm{GL}(n,k) :\ \det X = 1\}$$

is a topological group as a subgroup of $\mathrm{GL}(n,k)$. We can thus define subgroups of all of the above linear groups by taking the intersection with $\mathrm{SL}(n,k)$ with the appropriate field. So for example

$$\begin{aligned}\mathrm{SO}(n,\mathbb{R}) &:= \mathrm{SL}(n,\mathbb{R}) \cap \mathrm{O}(n,\mathbb{R})\\ \mathrm{SO}(p,q) &:= \mathrm{SL}(p+q,\mathbb{R}) \cap \mathrm{O}(p,q)\\ \mathrm{SO}(n,\mathbb{C}) &:= \mathrm{SL}(n,\mathbb{C}) \cap \mathrm{O}(n,\mathbb{C})\\ \mathrm{SU}(n) &:= \mathrm{SL}(n,\mathbb{C}) \cap \mathrm{U}(n)\,.\end{aligned}$$

Notice that the subgroup N in Example 2.11 is in $\mathrm{SL}(n,\mathbb{R})$. Moreover $A := A_{\det} \cap \mathrm{SL}(n,\mathbb{R})$ is also an important non-trivial subgroup of $\mathrm{SL}(n,\mathbb{R})$.

Example 2.15 Let $\mathcal{H}$ be a complex separable Hilbert space (see Definition A.8). The space of unitary operators of $\mathcal{H}$

$$\begin{aligned}\mathcal{U}(\mathcal{H}) :=& \{U\colon \mathcal{H} \to \mathcal{H} :\ U^{-1} = U^*\}\\ =& \{U\colon \mathcal{H} \to \mathcal{H} :\ UU^* = U^*U = \mathrm{Id}\}\end{aligned}$$

is a topological group with the strong operator topology.

2.2 Compactness and Local Compactness

The Examples 2.1, 2.2 and 2.3 are obviously locally compact. Likewise the Examples 2.4, 2.11 and 2.12 are also locally compact because of Lemma A.2, as well as Example 2.5 if $\mathbb{F}$ is

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locally compact.

Example 2.16 (See Example 2.6) The homeomorphism group of a topological space X is not necessarily locally compact, even if X is compact **(see Exercise 3.)**.

Example 2.17 (See Example 2.8) Contrary to the homeomorphism group, the isometry group of a metric space X is as "good" as the space itself. In other words, if X is compact, then $\mathrm{Iso}(X)$ is compact and if X is locally compact, then $\mathrm{Iso}(X)$ is locally compact **(Exercise 4.)**. So $\mathrm{Iso}(X)$ is always much much smaller than $\mathrm{Homeo}(X)$.

The proof of the first assertion follows immediately from Ascoli–Arzelà's Theorem (see Theorem A.1). In fact, from the chain of inclusions

$$\mathrm{Iso}(X) \subset \mathrm{Homeo}(X) \subset C(X, X)$$

it follows that $\mathrm{Iso}(X)$ is compact if it is an equicontinuous totally bounded family, which is obvious since it consists of isometries and X is compact.

Example 2.18 (See Example 2.11.3. and 2.12) We mentioned already that $\mathrm{O}(p, q)$ is locally compact since it is a (closed) subgroup of $\mathrm{GL}(p+q, \mathbb{R})$. The question now is whether it is compact and we will show that $\mathrm{O}(p, q)$ is compact if and only if $p = 0$ or $q = 0$.

1. Let $p = 0$ and let $\mathrm{O}(0, n) = \mathrm{O}(n, \mathbb{R})$. Let us write $A \in \mathrm{O}(n, \mathbb{R})$ as $A = ((c_1), \dots, (c_n))$, where $c_j = Ae_j$ for $1 \leq j \leq n$. Thus $\{c_1, \dots, c_n\}$ is an orthonormal basis in $\mathbb{R}^n$. In particular $\|c_j\|^2 = 1$, so that $|A_{ij}| \leq 1$. Thus $\mathrm{O}(n, \mathbb{R})$ is bounded in $\mathbb{R}^{n\times n}$. On the other hand, by definition
$$\mathrm{O}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \langle Av, Aw\rangle = \langle v, w\rangle = \text{ for all } v, w \in \mathbb{R}^n\} =^{-1} (\mathrm{Id}) \,,$$
where $f \colon M_{n\times n} \to M_{n\times m}$ is defined as $f(A) := A^t A$, so that $\mathrm{O}(n, \mathbb{R})$ is closed and hence compact by the Heine–Borel Theorem.
2. Let us assume now that $pq \neq 0$ and we will show that in this case $\mathrm{O}(p, q)$ is not compact since it is not bounded. In fact we can write for $n = p + q$
$$\mathrm{O}(p, q) := \{A \in \mathbb{R}^{n\times n} : Q_p(Av) = Q_p(v) \text{ for all } v \in \mathbb{R}\} \,,$$
where
$$Q_p(v) = -\sum_{j=1}^{p} v_j^2 + \sum_{j=p+1}^{n} v_j^2 \,.$$

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Consider for example the case $p = 1$, so that

$$Q_1(v) = -v_1^2 + \sum_{j=2}^{n} v_j^2$$

with respect to the basis $(e_1, \ldots, e_n)$. Consider now the change of basis

$$\begin{aligned} e_1' &:= e_2 - e_1 \\ e_2' &:= e_2 + e_1 \\ e_j' &:= e_j \text{ for } 3 \le j \le n\,, \end{aligned}$$

and denote by V the vector space $\mathbb{R}^n$ with this new basis. On V the quadratic form will now take the form

$$Q_1'(v') = -(v_1' - v_2')^2 + (v_1' + v_2')^2 + \sum_{j=3}^{n} {v_j'}^2\,.$$

The matrix

$$A_s := \begin{pmatrix} s & 0 & {}^t0_{n-2} \\ 0 & \frac{1}{s} & {}^t0_{n-2} \\ 0_{n-2} & 0_{n-2} & \mathrm{Id}_{n-2} \end{pmatrix} \tag{2.4}$$

clearly satisfies

$$Q_1'(A_s v) = Q_1'(v)$$

so that $A_s \in \mathrm{O}(V, Q_1')$. Moreover one can show that the subgroup $\{A_s : s > 0\}$ is closed, which shows that $\mathrm{O}(V, Q_1')$ is not compact. The general argument for $n > p \ge 1$ is analogous.

Example 2.19 The special linear group $\mathrm{SL}(n, \mathbb{R})$ is a locally compact group since it is closed in $\mathrm{GL}(n, \mathbb{R})$, but it is not compact since the matrix A_t in (2.4) belongs to $\mathrm{SL}(n, \mathbb{R})$ as well.

Example 2.20 (See Example 2.10) Profinite groups are compact.

Example 2.21

1. The *one-dimensional torus*

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

with the usual multiplication is a compact Abelian topological group isomorphic to

$$\mathrm{SO}(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

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via the isomorphism

$$\begin{aligned} \mathrm{SO}(2,\mathbb{R}) &\longrightarrow \mathbb{T} \\ X &\longmapsto e^{\imath\vartheta}\,. \end{aligned}$$

2. The *n-dimensional torus* $\mathbb{T}^n$ is also a compact Abelian topological group.

Example 2.22 We emphasise that $\mathrm{U}(n) \neq \mathrm{O}(n,\mathbb{C})$. In fact:

- $\mathrm{U}(n)$ preserves the usual Hermitian inner product on $\mathbb{C}^n$, Thus

$$\mathrm{U}(n) = \{X \in \mathrm{GL}(n,\mathbb{C}) :\ {}^*XX = \mathrm{Id}_n\}\,,$$

where ${}^*X = {}^t\overline{X}$ and it is compact.

- $\mathrm{O}(n,\mathbb{C})$ preserves a non-degenerate symmetric bilinear form on $\mathbb{C}^n$ so that

$$\mathrm{O}(n,\mathbb{C}) := \{X \in \mathrm{GL}(n,\mathbb{C}) :\ {}^tXX = \mathrm{Id}_n\}$$

and $\mathrm{O}(n,\mathbb{C})$ is not compact for $n \geq 2$. The argument to see this is exactly the same as for $\mathrm{O}(p,q)$.

Example 2.23 Let $B\colon \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ be the skew-symmetric bilinear form on $\mathbb{C}^{2n}$ given by $B(x,y) = \sum_{1\leq p\leq n} x_p y_{n+p} - x_{n+p} y_p$, where $x = (x_1,\dots,x_{2n})$ and $y = (y_1,\dots,y_{2n})$. The symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ is defined as the subgroup of $\mathrm{GL}(2n,\mathbb{C})$ of matrices that leave B invariant. If $F = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, then

$$\begin{aligned} \mathrm{Sp}(2n,\mathbb{C}) &:= \{A \in \mathrm{GL}(2n,\mathbb{C}) : B(x,y) = B(Ax,Ay) \text{ for all } x,y \in \mathbb{C}^{2n}\} \\ &= \{A \in \mathrm{GL}(2n,\mathbb{C}) : {}^tAFA = F\}\,. \end{aligned}$$

Related to $\mathrm{Sp}(2n,\mathbb{C})$ there are also the following groups

$$\mathrm{Sp}(2n) := \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{U}(2n)\,,$$

which is compact, and

$$\mathrm{Sp}(2n,\mathbb{R}) := \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{GL}(2n,\mathbb{R}) = \{A \in \mathrm{GL}(2n,\mathbb{R}) : {}^tAFA = F\}.$$

Example 2.24 We get back to the space of unitary operators of a complex separable Hilbert space $\mathcal{H}$.

Lemma 2.2

If $\mathcal{H}$ is a complex separable Hilbert space, then the space of continuous unitary operators $\mathcal{U}(\mathcal{H})$ is a topological group that is locally compact if and only if $\dim\mathcal{H} < \infty$, in which

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case it is compact. ♡

Proof (⇐) Let us assume that $\dim \mathcal{H} = n < \infty$. Then

$$\mathcal{U}(\mathcal{H}) = \mathrm{U}(n)\,,$$

which is compact.

(⇒) We prove the assertion by contradiction. A basis neighborhood of $\mathrm{Id} \in \mathcal{U}(\mathcal{H})$ in the strong operator topology is of the form

$$U_{F,\epsilon} := \{T \in \mathcal{U}(\mathcal{H}) :\ \|Tu - u\| < \epsilon \text{ for all } u \in F\}\,,$$

where $F \subset \mathcal{H}$ is a finite set and $\epsilon > 0$. If $\mathcal{U}(\mathcal{H})$ is locally compact, the neighborhood $U_{F,\epsilon}$ is contained in a compact set C. We will show that the assumption that $\mathcal{H}$ is infinite dimensional leads to a contradiction.

We write $\mathcal{H} = \langle F\rangle \oplus \langle F\rangle^{\perp}$. Then an obvious verification shows that the subgroup

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{U}(\langle F\rangle^{\perp}) \end{pmatrix} \subset U_{F,\epsilon}\,,$$

since if $T \in \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{U}(\langle F\rangle^{\perp}) \end{pmatrix}$, then $Tu = u$ for all $u \in F$. But then

$$\mathcal{U}(\langle F\rangle^{\perp}) \simeq \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathcal{U}(\langle F\rangle^{\perp}) \end{pmatrix} \subset U_{F,\epsilon}\,,$$

that is also $\mathcal{U}(\langle F\rangle^{\perp})$ must be contained in a compact set and hence be compact. But if $F \subset \mathcal{H}$ is finite and $\mathcal{H}$ is infinite dimensional, then $\langle F\rangle^{\perp}$ must be infinite dimensional. We show now that the unitary group of an infinite dimensional Hilbert space cannot be compact.

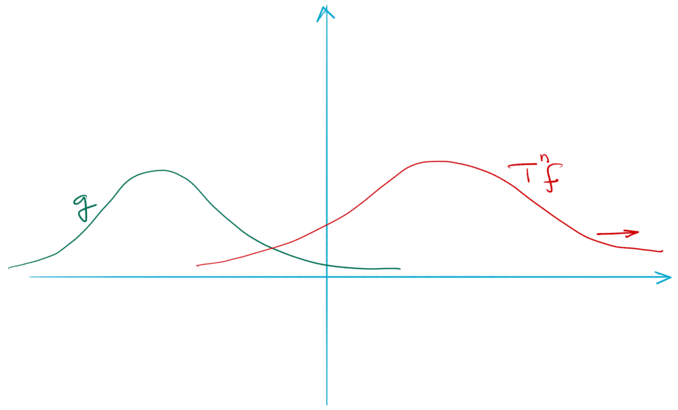
Claim 2.2.1. *If $\mathcal{H}$ is an infinite dimensional separable Hilbert space, then $\mathcal{U}(\mathcal{H})$ cannot be compact.*

By contradiction let us assume that $\mathcal{U}(\mathcal{H})$ is compact in the strong operator topology. If we can find a sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{U}(\mathcal{H})$ of unitary operators converging to zero in the weak operator topology, that is such that $\langle T_n u, v\rangle \to 0$ for all $u, v \in \mathcal{H}$, then by our assumption we could find subsequence $(T_{n_k})_{k\in\mathbb{N}}$ that converges in the strong operator topology to a unitary operator T. On the other hand $(T_{n_k})_{k\in\mathbb{N}}$ converges weakly to zero, which implies that $T = 0$, a contradiction.

Thus, in order to complete the proof, we need to show that if $\mathcal{H}$ is infinite dimensional and separable we can find a sequence of unitary operators $(T_n)_{n\in\mathbb{N}}$ that converges to zero in

-

the weak operator topology. Let $\mathcal{H} = L^2(\mathbb{R})$ and let $T \in L^2(\mathbb{R})$ be the translation by one, $Tf(x) := f(x-1)$. Then if $f, g \in L^2(\mathbb{R})$, $\langle T^n f, g\rangle \to 0$. In fact, the space of C^∞ compactly supported functions on $\mathbb{R}$ is dense in $L^2(\mathbb{R})$ and $\langle T^n f, g\rangle$ is very small as soon as n is large enough that the supports of $T^n f$ and of g are almost disjoint.



□

2.3 General Facts about Topological Groups

The simple fact of requiring that the group operations are continuous has a plethora of interesting consequences, of which we illustrate here the most important ones.

Definition 2.2. Symmetric neighborhood

A neighborhood U of the identity $e \in G$ in a topological group is symmetric *if $g^{-1} \in U$ whenever $g \in U$.* ♣

Proposition 2.1

Let G be a topological group. Then

1. *If V is a neighborhood of the identity $e \in G$, there exists a symmetric neighborhood U of the identity contained in V.*
2. *If V is a neighborhood of the identity $e \in G$, there exists a symmetric neighborhood U of the identity such that $U^2 = U^{-1}U \subset V$.*
3. *If $H < G$ is a subgroup, then its closure $\overline{H}$ is also a subgroup*
4. *If G is connected any discrete normal subgroup is central.*

-

5. *The connected component G° of the identity is a closed normal subgroup.*
6. *Every open subgroup is closed.*
7. *If G is connected and U is any neighborhood of the identity $e \in G$, then $G = \cup_{n=1}^{\infty} U^n$.* ♠

Note that the converse of Proposition 2.1.6. is not true. For example $\mathbb{R} < \mathbb{R}^2$ is a closed subgroup that is not open.

Proof (1) is immediate by taking $U := V \cap V^{-1}$ and (2) is also immediate from the continuity of the multiplication and from (1).

(3) Since the multiplication and the inversion are continuous, then

$$m(\overline{H} \times \overline{H}) = m(\overline{H \times H}) \subseteq \overline{m(H \times H)} = \overline{H}$$
$$i(\overline{H}) \subseteq \overline{H}\,.$$

(4) Let D be a discrete normal subgroup and, for $h \in D$ fixed, let us define the continuous map $\gamma_h \colon G \to D$ by $\gamma_h(g) := ghg^{-1}$. We want to show that $\gamma_h(g) \equiv h$ and this will follow from the connectedness of G and the discreteness of D. In fact, since G is connected, γ_h is continuous and D is discrete, then $\mathrm{image}(\gamma_h)$ must be one point. Since $\gamma_h(e) = ehe^{-1} = h$, then $\gamma_h(g) = h$ for all $g \in G$. Thus $ghg^{-1} = h$ for all $g \in G$, so that $gh = hg$ for all $g \in G$, that is D is central.

(5) Let G° be the connected component of the identity $e \in G$. Since the multiplication $m \colon G^\circ \times G^\circ \to G$ is continuous and $G^\circ \times G^\circ$ is connected, then $m(G^\circ \times G^\circ)$ is connected. But $e \in m(G^\circ \times G^\circ)$, so that $m(G^\circ \times G^\circ) \subset G^\circ$, that is G° is closed under multiplication. Likewise the image of $i \colon G^\circ \to G^\circ$ is connected and contains e, so that $i(G^\circ) \subset G^\circ$. Thus G° is a group.

To see that G° is closed, observe that $G^\circ \subset \overline{G^\circ}$. But $\overline{G^\circ}$ is connected and contains the identity in G so that $\overline{G^\circ} \subset G^\circ$. Thus $\overline{G^\circ} = G^\circ$.

If $g \in G$, consider now the continuous map defined by the conjugation $c_g \colon G^\circ \to G$, $c_g(h) = ghg^{-1}$. Since G° is connected, $c_g(G^\circ)$ is connected, hence contained in G°, which means that G° is normal.

(6) Let $H < G$ be an open subgroup. If $L_g \colon G \to G$ is the left multiplication by $g \in G$, by continuity of the multiplication $L_g H$ is also open for all $g \in G$. Thus the union $G \setminus H = \cup gH$ over all $g \in G \setminus H$ is open and hence H is closed.

(7) Obviously $\cup_{n=1}^{\infty} U^n \subseteq G$. Let $V \subset U$ be an open symmetric neighborhood of $e \in G$ such that

-

$V^2 = V^{-1}V \subset U$. Then $H := \cup_{n=1}^{\infty} V^n \subseteq \cup_{n=1}^{\infty} U^n \subseteq G$ is an open subgroup of G, hence closed by (6). Since G is connected, we have equality. □

2.4 Local homomorphisms

The content of this section will be heavily used in the correspondence between Lie groups and Lie algebras presented in § **??** and it is of independent interest.

Definition 2.3. Local homomorphism

Let G, H be topological groups.

1. A **local homomorphism** *is a continuous map $\varphi : U \to H$, where U is a neighborhood of $e \in G$, such that whenever $x, y, xy \in U$*
$$\varphi(xy) = \varphi(x)\varphi(y).$$
2. *A local homomorphism $\varphi \colon U \to H$ is a* **local isomorphism** *if it is bijective onto $\varphi(U)$ and $\varphi^{-1} \colon \varphi(U) \to G$ is continuous.*

♣

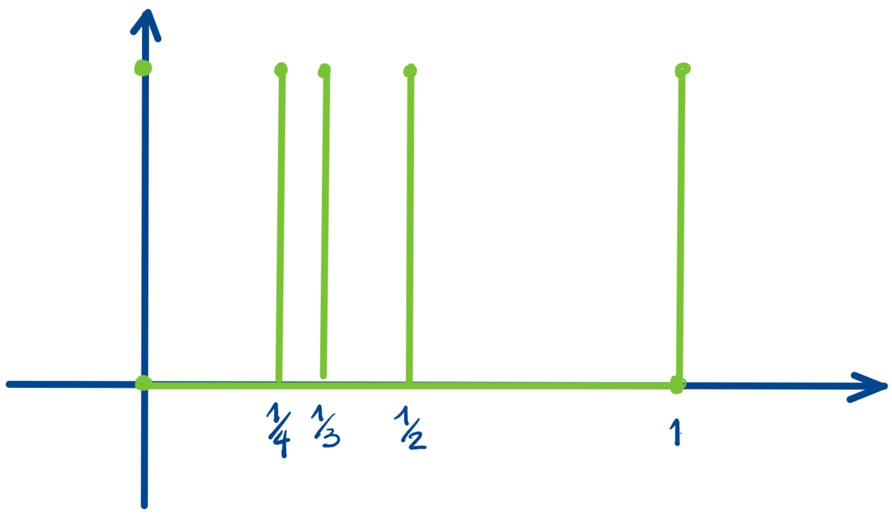
A natural question to ask is when a local homomorphism φ of a topological group can be extended to a homomorphism.

Theorem 2.1. Extension of local homomorphisms

If G is a simply connected topological group, then any local homomorphism extends uniquely to a homomorphism $G \to H$.

♡

Recall that a topological space X is *simply connected* if it is path-connected and $\pi_1(X)$ is trivial. Path-connectedness implies connectedness but the converse is not true in general. For example, let $X := [0,1] \times \{0\} \cup \{\{\frac{1}{n}\} \times [0,1] : n \in \mathbb{N}\} \cup \{\{0\} \times [0,1] \setminus \{0\} \times (0,1)\}$.



Then X is connected but not path-connected.

However, if a space is connected and locally path-connected, then it is path-connected. For example connected manifolds, and in particular connected Lie groups, are automatically path-connected, since they are locally homeomorphic to $\mathbb{R}^n$, which is path-connected.

Proof We give only the sketch of the proof. For the complete argument see [4]. Let $U \subset G$ be a neighborhood of $e \in G$ and $\varphi \colon U \to H$ the local homomorphism that we want to extend. We will prove the theorem in three steps:

1. We use that G is path-connected to define φ on all of G.
2. We use that $\pi_1(G) = 0$ to show that the extension is well-defined.
3. We show that φ is the unique continuous extension of $\varphi|_U$.

1. Since G is path-connected, if $g \in G$, let $\alpha : [0,1] \to G$ be a path from e to g. Choose a partition of $[0,1]$ into subintervals $I_k := [t_{k-1}, t_k]$, for $k = 1, \dots, n$, with the property that if $s, t \in I_k$, then

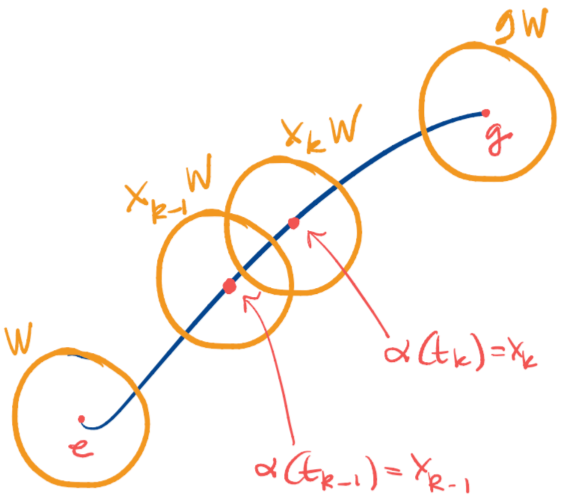
$$\alpha(s)^{-1}\alpha(t) \in U\,.$$

We call such a partition *good*. We impose a further condition that will be needed only in Step 2., but that we may as well impose from the beginning. We choose W be a neighborhood of $e \in G$ contained in U such that $W = W^{-1}$ and $W^2 \subset U$ and $\alpha \subset \cup_{k=0}^{n} \alpha(t_k)W$. Such a partition exists since $[0,1]$ is compact and the group operations are continuous, so that there exists a $\delta > 0$ such that $\alpha(s)^{-1}\alpha(t) \in U$ whenever $|s - t| < \delta$. Set $x_k := \alpha(t_k) \in \alpha$, with $x_0 = \alpha(0) = e$ and

$g = \alpha(t_n) = x_n$. Then

$$g = (x_0^{-1}x_1)(x_1^{-1}x_2)\dots(x_{n-1}^{-1}x_n)\,,$$

with $x_{k-1}^{-1}x_k \in U$.



Then we define

$$\varphi_\alpha(g) := \varphi(x_0^{-1}x_1)\varphi(x_1^{-1}x_2)\dots\varphi(x_{n-1}^{-1}x_n)\,,$$

To show that $\varphi_\alpha(g)$ is independent of the partition, notice that adding points to the partition gives a partition that still has the above defining properties. Let us hence take $t \in I_k$ and write $[t_{k-1}, t_k] = [t_{k-1}, t] \cup [t, t_k]$. Since $t \in I_k$, then $\alpha(t_{k-1})^{-1}\alpha(t) \in U$, $\alpha(t)^{-1}\alpha(t_k) \in U$ and

$$\alpha(t_{k-1})^{-1}\alpha(t_k) = \alpha(t_{k-1})^{-1}\alpha(t)\alpha(t)^{-1}\alpha(t_k) \in U\,,$$

so that

$$\varphi_\alpha(\alpha(t_{k-1})^{-1}\alpha(t_k)) = \varphi(\alpha(t_{k-1})^{-1}\alpha(t))\varphi(\alpha(t)^{-1}\alpha(t_k))\,.$$

2. We show now that $\varphi_\alpha(g)$ is in fact independent of α. Since $\pi_1(G) = 0$ we can choose a homotopy $H\colon [0,1] \times [0,1] \to G$ with $H(0,t) = \alpha_0(t)$ and $H(1,t) = \alpha_1(t)$ and set $\varphi_s := \varphi_{\alpha_s}$, where $\alpha_s\colon [0,1] \to G$ is defined as $\alpha_s(t) := H(s,t)$. Let $\delta > 0$ be such that

$$H(s_1,t_1)^{-1}H(s_2,t_2) \in W$$

for all $s_1, s_2, t_1, t_2 \in [0,1]$ with $|s_1 - s_2| + |t_1 - t_2| < \delta$. Then for all $s \in [0,1]$, the partition $\{t_k\}_{k=0}^n := \{\frac{k}{n}\}_{k=0}^n$ is good, where we choose n large enough that $\frac{1}{n} < \frac{\delta}{2}$.

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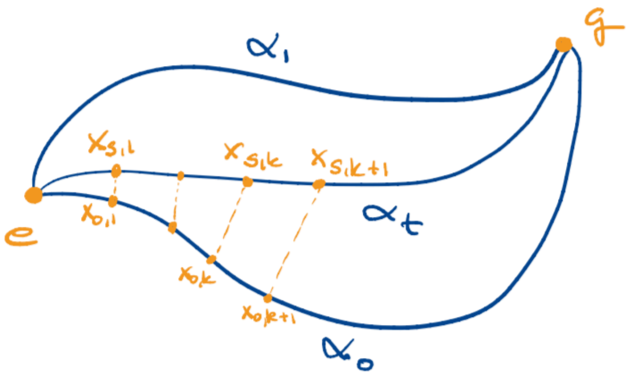


Let

$$A := \{s \in [0,1] : \varphi_s(g) = \varphi_0(g)\}.$$

Since $0 \in A \neq \varnothing$, it will be enough to show that A is open and closed.

To see that A is closed, we will show that if $(s_j)_{j\in\mathbb{N}} \subset A$ and $s_j \to s$ for $j \to \infty$, then $s \in A$. Let α_{s_j} and α_s be the corresponding paths and let $\{t_k\}_{k=0}^n$ be the good partition of $[0,1]$ chosen above. By continuity of H, one deduces that

$$\lim_{j\to\infty} \alpha_{s_j}(t_k) = \alpha_s(t_k).$$

Writing $x_{s_j,k} := \alpha_{s_j}(t_k)$ and $x_{s,k} := \alpha_s(t_k)$, and using that the φ_{s_j} are continuous, one deduces that

$$\lim_{j\to\infty} \varphi(x_{s_j,k-1}^{-1} x_{s_j,k}) = \varphi_s(x_{s,k-1}^{-1} x_{s,k}).$$

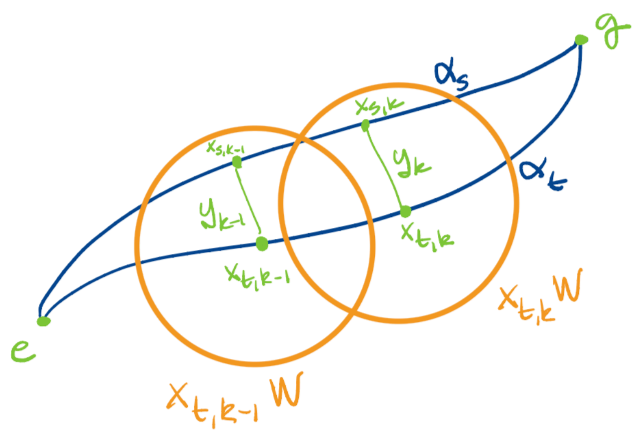
Thus

$$\lim_{j\to\infty} \varphi_{s_j}(g) = \lim_{j\to\infty} \prod_{k=0}^{n-1} \varphi(x_{s_j,k}^{-1} x_{s_j,k+1}) = \prod_{k=0}^{n-1} \varphi_s(x_{s,k}^{-1} x_{s,k+1}).$$

Since each term on the left hand side is equal to $\varphi_0(g)$, so is the one on the right hand side.

To see that A is open, let $t \in A$ and let $s \in [0,1]$ be close enough to t so that $\alpha_s \subset \cup_{j=0}^n x_j W$, where $x_j := \alpha(t_j)$ were defined at the beginning of the proof. We can define $y_k := x_{s,k}^{-1} x_{t,k} \in W$ so that

$$x_{s,k-1}^{-1} x_{s,k} = y_{k-1} x_{t,k-1}^{-1} x_{t,k} y_k^{-1},$$



and

$$\varphi_s(g) = \prod_{k=1}^{n} \varphi(x_{s,k-1}^{-1} x_{s,k}) = \prod_{k=1}^{n} \varphi(y_{k-1} x_{t,k-1}^{-1} x_{t,k} y_k^{-1}) = \prod_{k=1}^{n} \varphi(x_{t,k-1}^{-1} x_{t,k}) = \varphi_t(g)\,.$$

Thus $s \in A$, that is A is open.

3. It is easy to see that φ is continuous. To see that it is a homomorphism, let α be a path from e to g and β a path from e to h. Then the concatenation of α with $g\beta$ is a path from e to gh and by definition $\varphi(gh) = \varphi(g)\varphi(h)$. The uniqueness follows immediately from Proposition 2.1.7. □

2.5 Haar Measure and Homogeneous Spaces

2.5.1 Haar Measure

Let X be a locally compact topological space and G a topological group. A *left action* of G on X by homeomorphisms is a homomorphism $G \to \mathrm{Homeo}(X)$, that is a map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \end{aligned}$$

such that $(g_2 g_1)x = g_2(g_1 x)$ for all $g_1, g_2 \in G$ and $x \in X$. The action is continuous if $G \times X \to X$ is a continuous map, in which case

$$\begin{aligned} \varphi_g \colon X &\longrightarrow X \\ x &\longmapsto gx \end{aligned}$$

is a homeomorphism with inverse $\varphi_{g^{-1}}$. If $C_c(X)$ is the space of continuous functions with compact support and G acts on X, then there is a continuous representation $\lambda\colon G \to \mathrm{Iso}(C_c(X))$

-

defined by $(\lambda(g)f)(x) := f(g^{-1}x)$ (see Lemma A.3). Likewise G acts continuously on the left on the space $C_c(X)^*$ of continuous linear functionals on $C_c(X)$, via the contragredient representation $\lambda^*(g)(\Lambda))(f) := \Lambda(\lambda(g^{-1})f)$.

Remark A right G-action on X $(g,x) \longmapsto xg$ would induce a right G-action on $C_c(X)$, $(\rho(g)f)(x) := f(xg)$ and hence on $C_c(X)^*$, $(\rho^*(g)(\Lambda))(f) := \Lambda(\rho(g)(f))$.

A left (resp. right) action is an action for which, given the product g_1g_2 acting on X, first g_2 acts (resp. g_1) followed then by g_1 (resp. g_2).

Theorem 2.2. Riesz Representation Theorem

Let X be a locally compact Hausdorff topological space. If Λ is a positive linear functional on $C_c(X)$ (that is $\Lambda(f) \geq 0$ if $f \in C_c(X)$ with $f \geq 0$), then there exists a unique regular Borel measure μ on X that represents Λ, that is such that for every $f \in C_c(X)$,

$$\Lambda(f) = \int_X f(x)\, d\mu(x)\,.$$

♡

(For the definition of *regular Borel measure* see Definition A.7.)

Notice that the action on the left of a group G on Λ is reflected in the action on the measure given by the contragredient action and the identification of functionals with regular Borel measures given by Riesz Representation Theorem. In other words the G-action on measures on $C_c(X)$ is denoted by $(g,\mu) \mapsto g_*\mu$, where

$$(g_*\mu)(A) := \mu(g^{-1}A)\,,$$

so that

$$(\lambda(g)^*\Lambda)(f) = \int_X f(gx)\, d\mu(x) = \int_X f(x)\, d(g_*\mu)(x) = \int_X f(x)\, d\mu(g^{-1}x)\,.$$

A particular action is the one of a locally compact Hausdorff group on itself.

Definition 2.4. (Haar measure)

A left *(resp.* right*)* Haar measure *on a locally compact Hausdorff group G is a non-zero positive linear functional*

$$m \colon C_c(G) \to \mathbb{C}$$

that is invariant under left (resp. right) translation, that is such that

$$(g_*m)(f) = m(f)$$

-

for all $f \in C_c(G)$. ♣

In the following we will use the notations $m(f)$, $\int_G f(x)$, $dm(x)$ or dx according to what we want to emphasize or for simplicity.

Theorem 2.3. (Existence and Uniqueness of the Haar measure, 1933)

A left (resp. right) Haar measure on a locally compact Hausdorff group always exists and is unique up to positive multiplicative constants. ♡

We will verify the uniqueness. However the proof of the existence of the Haar measure in general is long, technical and does not bring much insight. There are however cases in which the proof is simple and follows on standard yet useful techniques. This is the case for example for compact groups (see [13, Theorem 2.2.3]) or for Lie groups **(see ??)**.

Lemma 2.3

Let m be a left Haar measure. If $f \in C_c(G)$ and $x \in G$, let $\check{f}(x) := f(x^{-1})$. Then $n(f) := m(\check{f})$ is a right Haar measure. ♡

Proof We need to verify that $n(\rho(g)(f)) = n(f)$ for every $g \in G$ and for every $f \in C_c(G)$. Notice that

$$(\rho(g)f)\check{}(x) = (\rho(g)f)(x^{-1}) = f(x^{-1}g)$$

so that

$$\begin{aligned} n(\rho(g)f) = m((\rho(g)f)\check{}) &= \int_G f(x^{-1}g)\, dm(x) \\ &= \int_G \check{f}(g^{-1}x)\, dm(x) = \int_G \check{f}(x)\, dm(x) = n(f)\,. \end{aligned}$$

□

Lemma 2.4

Let G be a locally compact Hausdorff group with left Haar measure m. Then

1. $\operatorname{supp}(m) = G$*, and*
2. *If $h \in C(G)$ is such that*
$$\int_G h(x)\varphi(x)\, dm(x) = 0$$
for all $\varphi \in C_c(G)$, then $h \equiv 0$. ♡

Proof (1) Recall that $\operatorname{supp}(m) := \{x \in G :$ for every open set U containing x, $m(U) > 0\}$. Since $m \not\equiv 0$, there exists $f \in C_c(G)$ such that $m(f) > 0$. Let $K := \operatorname{supp}(f)$ with $m(K) > 0$.

-

If $G \neq \operatorname{supp}(m)$, then there exists $x \in G \setminus \operatorname{supp}(m)$ and an open neighborhood $U \ni x$ with $m(U) = 0$. But a finite number of translates of U would cover K, so that $m(K) = 0$, which is a contradiction.

(2) We show that $h(e) = 0$ (which is anyway all we need in the proof of the uniqueness of the Haar measure) and the argument for any other point follows by translation. Let $\epsilon > 0$. By continuity of h there exists an open neighborhood $V \ni e$ such that for all $g \in V$

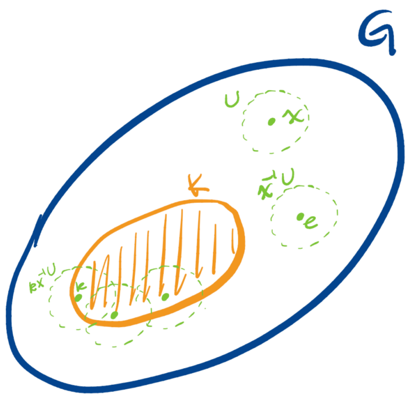
$$|h(g) - h(e)| < \epsilon\,.$$

By Urysohn's lemma there exists $\varphi \in C_c(G)$ such that $\varphi \geq 0$, $\varphi(e) > 0$ and $\operatorname{supp}(\varphi) \subset V$. Since $\int_G h(g)\varphi(g)\,dm(g) = 0$ for all $\varphi \in C_c(G)$, then

$$\begin{aligned}
&|h(e)|\left|\int_G \varphi(g)\,dm(g)\right| \\
&= \left|\int_G h(e)\varphi(g)\,dm(g)\right| \\
&= \left|\int_G h(g)\varphi(g)\,dm(g) - \int_G h(e)\varphi(g)\,dm(g)\right| \\
&\leq \int_G |h(g) - h(e)|\varphi(g)\,dm(g) \\
&\leq \epsilon \int_G \varphi(g)\,dm(g)\,,
\end{aligned}$$

from which it follows that $|h(e)| < \epsilon$ for all $\epsilon > 0$, that is $h(e) = 0$. □

We remark that we used in the first part of the proof that G is a topological group. In fact, the fact that we can cover K with translates of a neighborhood U of $x \in G \setminus K$ is only possible because we are in a topological group.



Proof [Proof of the uniqueness of the Haar measure in Theorem 2.3] Let m be an arbitrary left Haar measures and n an arbitrary right Haar measure (which exists by Lemma 2.3). Let $f, g \in C_c(G)$

-

be such that $m(f) \neq 0$ (this certainly exists since m is non-zero).

$$\begin{aligned} m(f)n(g) &= m(f)\int_G g(y)\,dn(y) \\ &\overset{(1)}{=} m(f)\int_G g(yt)\,dn(y) \\ &= \int_G f(t)\left(\int_G g(yt)\,dn(y)\right)dm(t) \\ &\overset{(2)}{=} \int_G\left(\int_G f(t)g(yt)\,dm(t)\right)dn(y) \\ &\overset{(3)}{=} \int_G\left(\int_G f(y^{-1}x)g(x)\,dm(x)\right)dn(y) \\ &\overset{(4)}{=} \int_G\left(\int_G f(y^{-1}x)\,dn(y)\right)g(x)\,dm(x)\,, \end{aligned}$$

where we used:

– in (1) that n is right invariant;

– in (2) and in (4) Fubini;

– in (3) the right invariance of n, the left invariance of m and we set $x = yt$.

Note that we could use Fubini's theorem since the support of all functions is compact and hence

$$\int_{G\times G} |f(t)g(yt)|\,dm(t)\,dn(y) < \infty$$

and

$$\int_{G\times G} |f(y^{-1}x)g(x)|\,dm(x)\,dn(y) < \infty$$

Let us now define $w_f\colon G \to \mathbb{R}$ by

$$w_f(x) := \frac{1}{m(f)}\int_G f(y^{-1}x)\,dn(y)\,,$$

so that

$$n(g) = \frac{1}{m(f)}\int_G\left(\int_G f(y^{-1}x)\,dn(y)\right)g(x)\,dm(x) = \int_G w_f(x)g(x)\,dm(x)\,.$$

Since the left hand side is independent of f, for all $f_1, f_2 \in C_c(X)$ with $m(f_i) \neq 0$, $i = 1, 2$, then

$$\int_G w_{f_1}(x)g(x)\,dm(x) - \int_G w_{f_2}(x)g(x)\,dm(x) = 0\,.$$

Since $w_{f_1} - w_{f_2}$ is continuous, by Lemma 2.4 $w_f(e)$ is independent of f, so that $w_f(e) = C$ for some $C \in \mathbb{R}$. Thus

$$m(f)C = m(f)w_f(e) = m(f)\frac{1}{m(f)}\int_G f(y^{-1})\,dn(y) = \int_G f(y^{-1})\,dn(y) = n(\check{f})\,.$$

-

If now we choose $n(f) := m'(\check{f})$, which is a well-defined left Haar measure by Lemma 2.3, then

$$m(f)C = m'(f)$$

for all $f \in C_c(G)$ such that $m(f) \neq 0$. □

Example 2.25

1. The Lebesgue measure on $(\mathbb{R}^n, +)$ is the left and right Haar measure.
2. The Lebesgue measure on $G := (\mathbb{R}_{>0}, \cdot)$ is neither left nor right invariant, but
$$f \mapsto \int_G f(x)\,\frac{dx}{x}$$
defines both the left and the right Haar measure on G.
3. If G is discrete, then the counting measure is both a left and a right Haar measure.

The above examples bring to the question as to when a left Haar measure is also right invariant. To approach this question let $\mathrm{Aut}(G)$ be the group of continuous invertible automorphisms of G with continuous inverse. Then $\mathrm{Aut}(G)$ acts on $C_c(G)$ on the left via

$$(\alpha \cdot f)(x) := f(\alpha^{-1}(x))$$

for $\alpha \in \mathrm{Aut}(G)$, $f \in C_c(G)$ and $x \in G$. If m is a left Haar measure on G, one can easily verify that the linear form

$$f \mapsto m(\alpha \cdot f)$$

is also a left Haar measure. In fact

$$\begin{aligned} m(\alpha \cdot \lambda(g)(f)) &= \int_G (\alpha \cdot \lambda(g)(f))(x)\,dm(x) = \int_G (\lambda(g)f)(\alpha^{-1}(x))\,dm(x) \\ &= \int_G f(\alpha^{-1}(g^{-1}x))\,dm(x) = \int_G (\alpha \cdot f)(x)\,dm(x) = m(\alpha \cdot f) \end{aligned}$$

Thus there exists a positive constant $\mathrm{mod}_G(\alpha)$ such that

$$m(\alpha \cdot f) = \mathrm{mod}_G(\alpha) m(f)\,. \tag{2.5}$$

Lemma 2.5

The function $\mathrm{mod}_G\colon \mathrm{Aut}(G) \to (\mathbb{R}_{>0}, \cdot)$ *is a homomorphism.* ♡

Proof Since $\mathrm{Aut}(G)$ acts on $C_c(G)$ on the left, then $(\alpha\beta)\cdot f = \alpha\cdot(\beta\cdot f)$. Then for all $f \in C_c(G)$

$$\begin{aligned} \mathrm{mod}_G(\alpha\beta)m(f) &= m((\alpha\beta)\cdot f) = m(\alpha\cdot(\beta\cdot f)) \\ &= \mathrm{mod}_G(\alpha)m(\beta\cdot f) \\ &= \mathrm{mod}_G(\alpha)\mathrm{mod}_G(\beta)m(f)\,. \end{aligned}$$

-

□

Let now consider the conjugation automorphism $\alpha = c_g$, for $g \in G$,

$$\begin{aligned} c_g \colon G &\to G \\ x &\mapsto gxg^{-1}\,, \end{aligned} \tag{2.6}$$

so that $(c_g \cdot f)(x) = f(g^{-1}xg)$. We use the notation $\Delta_G(g) := \mathrm{mod}_G(c_g)$ and we call $\Delta_G \colon G \to (\mathbb{R}_{>0}, \cdot)$ the *modular function of* G. Explicitly the formula (2.5) for $\alpha = c_g$ gives

$$\begin{aligned} \Delta_G(g)m(f) = m(c_g \cdot f) &= \int_G (c_g \cdot f)(x)\, dm(x) = \int_G f(g^{-1}xg)\, dm(x) \\ &= \int_G f(xg)\, dm(x) = m(\rho(g)f) \end{aligned}$$

so that

$$m(\rho(g)f) = \Delta_G(g)m(f)\,, \tag{2.7}$$

which shows that the modular function captures the extent to which a given left Haar measure fails to be right invariant.

Proposition 2.2

Let G be a locally compact Hausdorff topological group with left Haar measure m and let $\Delta_G \colon G \to \mathbb{R}_{>0}$ be its modular function. Then

1. *Δ_G is continuous and*
2. *for every $f \in C_c(G)$*
$$\int_G f(x^{-1})\Delta_G(x)\, dm(x) = \int_G f(x)\, dm(x)\,.$$

Proof (1) Since $\rho \colon G \to \mathrm{Iso}(C_c(G))$ is continuous when $\mathrm{Iso}(C_c(G))$ is given the strong operator topology, then

$$\lim_{x \to y} \|\rho(x)f - \rho(y)f\|_\infty = 0$$

for all $f \in C_c(G)$ and all $x, y \in G$. It follows that

$$0 = \lim_{x \to y} |m(\rho(x)f) - m(\rho(y)f)| = \lim_{x \to y} |m(f)|\, |\Delta_G(x) - \Delta_G(y)|\,,$$

that is Δ_G is continuous.

-

(2) Let us set $f^*(x) := f(x^{-1})\Delta_G(x)$ and let us observe that

$$\begin{aligned}(\lambda(g)f)^*(x) &= (\lambda(g)f)(x^{-1})\Delta_G(x) = f(g^{-1}x^{-1})\Delta_G(x)\\ &= \Delta_G(g)^{-1}f(g^{-1}x^{-1})\Delta_G(xg) = \Delta_G(g)^{-1}f^*(xg)\\ &= \Delta_G(g)^{-1}(\rho(g)f^*)(x)\,.\end{aligned}$$

Notice that $m'(f) := m(f^*)$ is also a left Haar measure. In fact,

$$m((\lambda(g)f)^*) = m(\Delta_G(g)^{-1}(\rho(g)f^*)) = \Delta_G(g)^{-1}m(\rho(g)f^*) = \Delta_G(g)^{-1}\Delta_G(g)m(f^*) = m(f^*)$$

Thus there exists $C > 0$ such that $m'(f) = Cm(f)$ and we want to show that $C = 1$. Since Δ_G is continuous, for every $\epsilon > 0$ there exists a symmetric neighborhood $V \ni e$ such that

$$|\Delta_G(x) - 1| < \epsilon$$

for every $x \in V$. Let $f \in C_c(G)$ be a symmetric function such that $f \geq 0$ and with support in V and such that $m(f) = 1$. Then for every $\epsilon > 0$

$$\begin{aligned}|1 - C| &= |(1 - C)m(f)| = |m(f) - m'(f)| = |m(f) - m(f^*)|\\ &\overset{(*)}{=} |m(f) - m(\Delta_G f) = |m((1 - \Delta_G)f)| < \epsilon\, m(f) = \epsilon\,,\end{aligned}$$

where in $(*)$ we used that f is symmetric and in $(**)$ that $f \geq 0$. □

Definition 2.5

A group G is unimodular *if $\Delta_G \equiv 1$, that is if the left Haar measure and the right Haar measure coincide.* ♣

Since for a left Haar measure m we have by Lemma 2.3 that $m(\check{f})$ is a right Haar integral, the following is immediate

Corollary 2.1

The Haar measure of a group G is inverse invariant if and only if the group is unimodular. ♡

Example 2.26

1. Any locally compact Hausdorf Abelian group is unimodular.
2. Any discrete group is unimodular, since the Haar measure is just the counting measure.
3. Since there are no non-trivial compact subgroups of $(\mathbb{R}_{>0}, \cdot)$, any compact group is unimodular.
4. We show that $\mathrm{GL}(n, \mathbb{R})$ is unimodular. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, we consider the restriction $dm(X) := \prod_{i.j=1}^n dX_{i,j}$ to $\mathrm{GL}(n, \mathbb{R})$ of the Lebesgue measure on $\mathbb{R}^{n\times n}$, where $X = (X_{i,j})_{i,j=1}^n$. We claim that $|\det X|^{-n}\, dm(X)$ is both a left and a right

-

Haar measure on $\mathrm{GL}(n,\mathbb{R})$. In fact, let $T_g\colon M_{n\times n}(\mathbb{R})\to M_{n\times n}(\mathbb{R})$ be defined by $T_g(X):=gX$ and let us observe, by writing $X=((v_1),(v_2),\dots,(v_n))$, that $|\det(dT_g)|\equiv|\det g|^n$. Thus

$$\begin{aligned}
&\int_{\mathrm{GL}(n,\mathbb{R})}(\lambda(g)f)(X)|\det X|^{-n}\,dm(X)\\
=&\int_{\mathrm{GL}(n,\mathbb{R})} f(g^{-1}X)|\det X|^{-n}\,dm(X)\\
=&|\det g|^{-n}\int_{\mathrm{GL}(n,\mathbb{R})} f(g^{-1}X)|\det(g^{-1}X)|^{-n}\,dm(X)\\
=&|\det g|^{-n}\int_{\mathrm{GL}(n,\mathbb{R})} f(X)|\det(X)|^{-n}||\det(dT_{g^{-1}}(X))|^{-n}\,dm(X)\\
=&|\det g|^{-n}\int_{\mathrm{GL}(n,\mathbb{R})} f(X)|\det(X)|^{-n}||\det g^{-1}|^{-n}\,dm(X)\\
=&\int_{\mathrm{GL}(n,\mathbb{R})} f(X)|\det X|^{-n}\,dm(X)\,.
\end{aligned}$$

A similar calculation shows the right invariance.

5. We consider the group $\mathbb{R}_{>0}\ltimes_\eta\mathbb{R}$, where $\eta\colon\mathbb{R}_{>0}\to\mathrm{Aut}(\mathbb{R})$ is defined by $\eta(a)(b):=ab$, so that the product is $(a,b)(a',b')=(aa',b+ab')$. Then $\mathbb{R}_{>0}\ltimes_\eta\mathbb{R}$ is the group of affine transformations of the real line, $(a,b)x=ax+b$, where $a\in\mathbb{R}_{>0}$ and $b\in\mathbb{R}$ and can be identified with the group

$$\left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} : a\in\mathbb{R}_{>0}, b\in\mathbb{R}\right\}$$

acting on $\mathbb{R}\simeq\{(x,0) : x\in\mathbb{R}\}\subset\mathbb{R}^2$. It is easy to verify that $\frac{da}{a^2}db$ is a left Haar measure and that $\frac{da}{a}db$ is a right Haar measure, so that $\mathbb{R}_{>0}\ltimes_\eta\mathbb{R}$ is not unimodular.

6. We consider the Heisenberg group $\mathbb{R}\ltimes_\eta\mathbb{R}^2$, where $\eta\colon\mathbb{R}\to\mathrm{Aut}(\mathbb{R}^2)$ is defined by

$$\eta(x)\begin{pmatrix}y\\ z\end{pmatrix}:=\begin{pmatrix}y\\ z+xy\end{pmatrix}$$

for $x\in\mathbb{R}$, $\begin{pmatrix}y\\ z\end{pmatrix}\in\mathbb{R}^2$, so that the group operation is

$$\left(x_1,\begin{pmatrix}y_1\\ z_1\end{pmatrix}\right)\left(x_2,\begin{pmatrix}y_2\\ z_2\end{pmatrix}\right)=\left(x_1+x_2,\begin{pmatrix}y_1+y_2\\ z_1+z_2+x_1y_2\end{pmatrix}\right).$$

-

It is easy to see that it can be identified with the group

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

and that the Lebesgue measure is both the left and the right Haar measure, so that $\mathbb{R} \ltimes_\eta \mathbb{R}^2$ is unimodular.

7. The group

$$P := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\} \tag{2.8}$$

is not unimodular since $\frac{da}{a^2} db$ is a left Haar measure and $da\, db$ is a right Haar measure.

8. Any closed normal subgroup of a unimodular group is unimodular. This follows from the following proposition.

Proposition 2.3

Let G be a locally compact Hausdorff group and let $H \trianglelefteq G$ be a closed normal subgroup. Then $\Delta_H = \Delta_G|_H$. Thus if G is unimodular, H is also unimodular. ♠

We will prove this later. For the moment we remark that it is essential that H is normal. In fact, for example $\mathrm{GL}(2, \mathbb{R})$ is unimodular, but the subgroup P in (2.8) is not.

Proposition 2.4

Let G be a locally compact Hausdorff topological group with left Haar measure m. Then $m(G) < \infty$ if and only if G is compact. ♠

Proof ($\Leftarrow$) Since G is compact, the function identically equal to 1 is in $C_c(G)$ Thus $m(G) = m(1) < \infty$.

($\Rightarrow$) Since m is regular and $m(G) < \infty$, there is a compact set $C \subset G$ with $m(C) < \frac{1}{2} m(G)$. But then since $m(xC) = m(C)$ for every $x \in G$, xC and C cannot be disjoint, hence $x \in CC^{-1}$, that is $G = CC^{-1}$, which is compact. □

If G is compact, the Haar measure is usually normalized so that $m(G) = 1$.

-

2.5.2 Homogeneous Spaces of Topological Groups

Let G be a group and $H < G$ a subgroup. Then G acts on the homogeneous space G/H on the left by translations $(g, g'H) \mapsto gg'H$ and the projection $p\colon G \to G/H$ is a G-map, that is it commutes with the G-action on G and on G/H. If G and H are topological groups, we endow G/H with the quotient topology, that is $U \subset G/H$ is open if and only if $p^{-1}(U) \subset G$ is open. This is the finest topology that makes p continuous.

Proposition 2.5

Let $H \leq G$ be topological groups. Then:

1. *The projection p is open, that is it sends open sets into open sets.*
2. *The action of G on G/H is continuous.*
3. *The quotient G/H is Hausdorff if and only if H is closed.*
4. *If G is locally compact, then also G/H is locally compact.*
5. *If G is locally compact and $H \leq G$ is closed, for every compact set $C \subset G/H$ there exists a compact set $K \subset G$ such that $p(K) = C$.*

Proof 1. and 2. follow from the definitions and the properties of topological groups.

3. If G/H is Hausdorff, then points are closed. In particular $eH \in G/H$ is closed and hence $p^{-1}(eH) = H \leq G$ is closed.

Conversely let us suppose that H is closed and let xH and yH be distinct points in G/H. Then xHy^{-1} is a closed set not containing the identity in G. Thus $G \smallsetminus xHy^{-1}$ is an open neighborhood of $e \in G$ and hence by Proposition 2.1 there exist U an open neighborhood of $e \in G$ such that $U^{-1}U \subset G \smallsetminus xHy^{-1}$. Thus $U^{-1}U \cap xHy^{-1} = \varnothing$, that is UxH and UyH are disjoint neighborhood respectively of xH and yH.

4. We have to show that every point in G/H has a compact neighborhood. Let $p(x) \in G/H$ and, since G is locally compact, let $x \in U \subset C$ with U open and C compact. Then $p(U)$ is open (by 1.), $p(C)$ is compact (since p is continuous) and $p(x) \in p(U) \subset p(C)$.

5. Let U be an open relatively compact neighborhood of $e \in G$. Then $\{p(Ux)\}_{x \in G}$ is an open cover of C and hence there exists a finite subcover $C \subset \cup_{j=1}^{n} p(Ux_j)$. Then

$$K := \bigcup_{j=1}^{n} \overline{U} x_j \cap p^{-1}(C) \subset G\,.$$

-

is a compact subset in G such that $p(K) = C$. □

If G acts transitively on a space X, then there is an isomorphism of G-spaces $G/G_x \to X$, where $G_x = \mathrm{Stab}_G(x)$ for $x \in X$, given by the map $gG_x \mapsto gx$. If X is a topological space and the action of G on X is continuous, then the G-map is also continuous. If G is a locally compact second countable Hausdorff space and X is locally compact Hausdorff, then the bijection is a homeomorphism.

Example 2.27

1. Let us consider the action of $\mathrm{O}(n+1,\mathbb{R})$ on $S^n \subset \mathbb{R}^{n+1}$. Notice that $g \in \mathrm{O}(n+1,\mathbb{R})$ if and only if ${}^t g g = \mathrm{Id}$, which implies that $\|gv\| = \|v\|$ for all $v \in \mathbb{R}^{n+1}$; in particular S^n is preserved by $\mathrm{O}(n+1,\mathbb{R})$. Moreover this action is transitive, that is $\mathrm{O}(n+1,\mathbb{R})e_{n+1} = S^n$ and in fact even the $\mathrm{SO}(n+1,\mathbb{R})$-action is transitive on S^n. The stabilizer of $e_{n+1} \in S^n$ is
$$\mathrm{SO}(n+1,\mathbb{R})_{e_{n+1}} = \{g \in \mathrm{SO}(n+1,\mathbb{R}) : ge_{n+1} = e_{n+1}\} \simeq \left\{\begin{pmatrix} \mathrm{SO}(n,\mathbb{R}) & 0 \\ 0 & 1 \end{pmatrix}\right\} < \mathrm{SO}(n+1,\mathbb{R}),$$
so that
$$S^n \simeq \mathrm{SO}(n+1,\mathbb{R})/\,\mathrm{SO}(n,\mathbb{R}).$$
2. The upper half plane $H^2_{\mathbb{R}} := \{x + \imath y \in \mathbb{C} : y > 0\}$ is an $\mathrm{SL}(2,\mathbb{R})$-space, with the $\mathrm{SL}(2,\mathbb{R})$-action given by *fractional linear transformations*: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $z \in H^2_{\mathbb{R}}$, then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}.$$
It is easy to see that the action is transitive since
$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix} \imath = x + \imath y$$
and that $\mathrm{SL}(2,\mathbb{R})_\imath = \mathrm{SO}(2,\mathbb{R})$. Thus the map $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(2,\mathbb{R}) \to H^2_{\mathbb{R}}$ identifies the upper half plane as the $\mathrm{SL}(2,\mathbb{R})$-orbit of $\imath$.
3. The group $\mathrm{SL}(2,\mathbb{R})$ acts transitively also on $\mathbb{R} \cup \{\infty\}$ with $P = \mathrm{SL}(2,\mathbb{R})_\infty$, where P is as in (2.8).
4. We generalize now the action in (2). Let
$$\mathrm{Sym}^+_1(n) := \{X \in M_{n\times n}(\mathbb{R}) : X \text{ is symmetric, positive definite and } \det(X) = 1\}.$$
Then $\mathrm{SL}(n,\mathbb{R})$ acts transitively on $\mathrm{Sym}^+_1(n)$ via $gX = gXg^t$, for $g \in \mathrm{SL}(n,\mathbb{R})$ and

$X \in \mathrm{Sym}_1^+(n)$. Moreover

$$\mathrm{SL}(n,\mathbb{R})_{Id_n} = \{g \in \mathrm{SL}(n,\mathbb{R}) : gId_ng^t = Id_n\} = \mathrm{SO}(n,\mathbb{R}),$$

so that

$$\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n,\mathbb{R}) \simeq \mathrm{Sym}_1^+(n).$$

If $n = 2$ this is nothing but the example in (2) (Exercise).

5. We generalize now the example in (3). We consider

$$\mathbb{P}^{n-1}(\mathbb{R}) = \mathbb{P}(\mathbb{R}^n) := \{V \subset \mathbb{R}^n : \text{ is a subspace with } \dim V = 1\}$$

with the transitive $\mathrm{SL}(n,\mathbb{R})$-action. In this case

$$\mathrm{SL}(n,\mathbb{R})_{<e_1>} = \left\{\begin{pmatrix} a & x \\ 0 & A\end{pmatrix} : a \in \mathbb{R}, a \neq 0, x \in \mathbb{R}^{n-1}, A \in \mathrm{GL}(n-1,\mathbb{R}), \det A = a^{-1}\right\}$$

and we identify $\mathrm{SL}(n,\mathbb{R})/\,\mathrm{SL}(n,\mathbb{R})_{<e_1>}$ with $\mathbb{P}^{n-1}(\mathbb{R})$. If $n = 2$ this is the example in (3).

6. Let

$$L := \{\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_n : f_j \in \mathbb{R}^n, \text{ for } j = 1,\dots,n, \det(f_1,\dots,f_n) = 1\}$$

be the space of lattices of covolume one in $\mathbb{R}^n$ (see Definition 2.7).

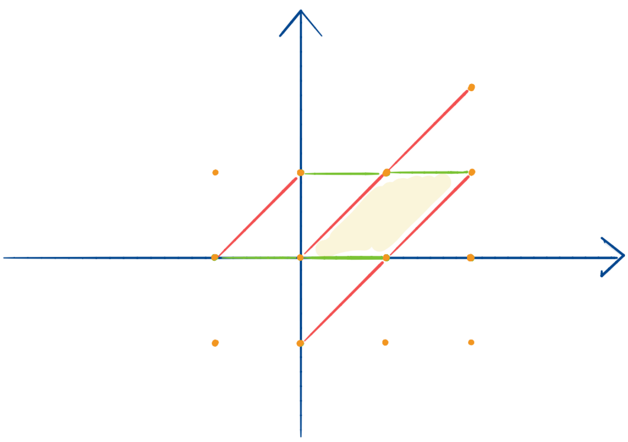


Figure 2.1: A lattice in L with $f_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $f_2 = \begin{pmatrix}1\\1\end{pmatrix}$

The group $\mathrm{SL}(n,\mathbb{R})$ acts transitively on L via

$$g(\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_n) := \mathbb{Z}gf_1 + \cdots + \mathbb{Z}gf_n$$

-

and the stabilizer of $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ is $\mathrm{SL}(n,\mathbb{Z})$. Thus L can be identified with $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$.

We prove now that if $H \trianglelefteq G$ is closed and normal, then $\Delta_G|_H = \Delta_H$. We start with the following lemma, only the first part of which (the definition) will be immediately used.

Lemma 2.6

Let G be a locally compact Hausdorff group and $H < G$ a closed subgroup. If $f \in C_c(G)$ and dh is the left Haar measure on H then

$$f^H(\dot{x}) := \int_H f(xh)\,dh$$

is in $C_c(G/H)$. Moreover the linear operator $A_H\colon C_c(G) \to C_c(G/H)$ defined as $A_H(f) := f^H$ is surjective. ♡

Proof The function f^H is obviously well defined as it is independent of the choice of representative of the coset xH. Moreover it is continuous[1]and $\operatorname{supp} f^H \subset p(\operatorname{supp} f)$. Thus $f^H \in C_c(G/H)$.

To prove the surjectivity, let $F \in C_c(G/H)$, let $C \subset G/H$ be the compact support of F and let $K \subset G$ be a compact set such that $p(K) = C$ (which exists by Proposition 2.5.5.). We will define $f \in C_c(G)$ such that $f^H = F$. Let $\eta \in C_c(G)$ such that $0 \le \eta \le 1$ and $\eta|_K \equiv 1$, which exists by Urysohn's Lemma ([7]). Then by definition

$$((F \circ p)\cdot \eta)^H = F \cdot \eta^H$$

so that $F = \frac{((F\circ p)\cdot\eta)^H}{\eta^H}$. Thus we define

$$f(g) := \begin{cases} \frac{(F\circ p)(g)\cdot \eta(g)}{(\eta^H(p))(g)} & \text{if } (\eta^H(p))(g) \neq 0 \\ 0 & \text{if } (\eta^H \circ p)(g) = 0\,, \end{cases}$$

which we need to verify to be in $C_c(G)$. In fact obviously $\operatorname{supp} f \subset \operatorname{supp} \eta$. Moreover f is continuous as it is continuous on two open sets U_1 and U_2 whose union is G, namely on

1. $U_1 := \{g \in G : (\eta^H \circ))(g) \neq 0\}$ by definition and on
2. $U_2 := G \setminus KH$, where it vanishes. If fact if $g \in G \setminus KH$, then $p(g) \notin C = \operatorname{supp} F$, so that $(F \circ p)(g) = 0$.

So the only thing to verify is that $G = U_1 \cup U_2$. In fact, if $g \in G$ and $g \notin U_1$, then

[1]A function $f\colon G \to \mathbb{C}$ is right (resp. left) uniformly continuous if for every $\epsilon > 0$ there exist a neighborhood V of $e \in G$ such that $|f(s) - f(t)| < \epsilon$ for every $ts^{-1} \in V$ (resp. $t^{-1}s \in V$). Right uniform continuity follows from Lemma **??** applied to $X = G$ and the action of G on G by left translations: an analogous statement holds for left uniform continuity.

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Appendix Preliminaries

p.131 (paragraph before definition A.11) The explanation for the coordinate chart of the tangent bundle is, if I am not mistaken, not correct. At least the definition that I know, is the one where the isomorphism $\mathbb{R}^n \to T_pM$ isn't any isomorphism, but the one induced by the coordinate chart (ϕ, U) that is $d\phi$.

A.1 Topological Preliminaries

We recall now a few well known concepts from topology.

Definition A.1. Basis of a topology

A basis $\mathcal{B}$ *of a topology* $\mathcal{T} \subset \mathcal{P}(X)$ *on a set* X *is a family* $\mathcal{B} \subset \mathcal{T}$ *such that every element of* $\mathcal{T}$ *is the union of elements of* $\mathcal{B}$*.* ♣

Example A.1 The family

$$\mathcal{B} := \{B_r(x) : r \in \mathbb{Q}_{\geq 0},\ x \in \mathbb{Q}^n\}$$

is a basis of the Euclidean topology on $\mathbb{R}^n$.

Lemma A.1. Characterization of a basis

Let X *be a set and* $\mathcal{T} \subset \mathcal{P}(X)$ *a topology. A family* $\mathcal{B} \subset \mathcal{T}$ *is a basis if and only if*

- $X = \cup_{Y \in \mathcal{B}} Y$*, and*
- *If* $B_1, B_2 \in \mathcal{B}$ *and* $B_1 \cap B_2 \neq \emptyset$*, then for every* $x \in B_1 \cap B_2$ *there exists* $B_3 \in \mathcal{B}$ *with* $x \in B_3 \subset B_1 \cap B_2$*.*

Then the topology is the family consisting of all possible unions of elements in $\mathcal{B}$*.* ♡

Definition A.2. Subbasis

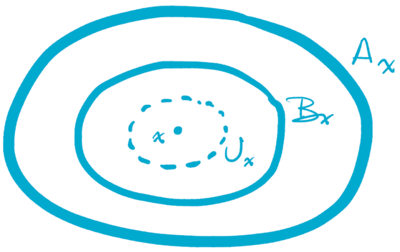
A subbasis $\mathcal{S}$ *of a topology* $\mathcal{T} \subset \mathcal{P}(X)$ *on a set* X *is a family of sets such that the family* $\mathcal{B}$ *obtained by taking all finite intersections of elements in* $\mathcal{S}$ *is a basis.* ♣

Definition A.3. Hausdorff topology

A topological space X is Hausdorff *if any two distinct points have disjoint neighborhood.* ♣

Definition A.4. Local Compactness

A topological space X is locally compact *if each point has a neighborhood basis consisting of compact sets, that is if for every $x \in X$ there exists a set $\mathcal{B}_x$ of compact neighborhoods of x such that any neighborhood A_x of x contains an element $B_x \in \mathcal{B}_x$.* ♣



Lemma A.2

Let X be a locally compact Hausdorff topological space. Every closed subset and every open subset of X is locally compact with respect to the induced topology. ♡

For any topological spaces X, Y one can define different topologies on the set

$$Y^X := \{f : X \to Y\},$$

or more specifically on the set

$$C(X,Y) : \{f : X \to Y : f \text{ is continuous}\}.$$

Definition A.5

Let X, Y be topological spaces.

- *The sets*

$$S(C,U) := \{f \in C(X,Y), f(C) \subset U\}$$

where $C \subset X$ is a compact set and $U \subset Y$ is an open set, form a subbasis of the compact-open *topology on $C(X,Y)$.*

- *The sets*

$$S(x,U) := \{f \in C(X,Y) : f(x) \in U\}$$

form a subbasis of the topology of the pointwise open *(or* pointwise convergence*) topology on $C(X,Y)$* ♣

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Remark Let X be a topological space and (Y, d) a metric space. The sets

$$B_C(f, \epsilon) := \{g \in C(X, Y) : \sup_{x \in C} d(f(x), g(x)) < \epsilon\},$$

where $C \subset X$ is a compact set, $\epsilon > 0$ and $f \in C(X, Y)$ form a basis of the compact-open topology. The set $B_C(f, \epsilon)$ consists of all functions $g \in C(X, Y)$ that are ϵ-close to f in all points in the compact set C. It is easy to see that if $\{f_n\} \subset C(X, Y)$, then $f_n \to f$ in the compact-open topology if and only if $f_n|_C \to f|_C$ uniformly on all compact sets $C \subset X$. In other words, if Y is a metric space the compact-open topology is nothing but the topology of the *uniform convergence on compact sets*.

In general the pointwise convergence is weaker than the uniform convergence on compact sets, which, in turn, is weaker that the uniform convergence. Of course the first two coincide on a set with the discrete topology and the last two on a compact set.

A.2 Functional Analytical Preliminaries

Theorem A.1. Ascoli–Arzelà's Theorem

Let (X, d_X) and (Y, d_Y) be compact metric spaces and let us consider the Banach space $C(X, Y)$ of continuous functions $f : X \to Y$ with the metric

$$d(f, g) := \sup_{x \in X} d(f(x), g(x)).$$

Let $\mathcal{F} \subset C(X, Y)$ be a subfamily of continuous functions. Then $\mathcal{F}$ is relatively compact if and only if it is equicontinuous, *that is for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) < \varepsilon$$

for every $f \in \mathcal{F}$, whenever $d_X(x, y) < \delta$. ♡

This is the form of the theorem that we need. Notice however that

- X need not be a metric space for the theorem to hold, and
- If Y is not compact then the theorem still holds, provided we add the assumption that the set $\{f(x) : f \in \mathcal{F}\}$ i s relatively compact for all $x \in X$.

If E, F are normed spaces, let us consider the normed space

$$\mathcal{B}(E, F) := \{T : E \to F : T \text{ is continuous and linear }\},$$

with $\|T\| := \sup_{\|x\|_E = 1} \|T(x)\|_F$.

-

If $T \in \mathcal{B}(E, F)$ is bijective and the inverse is continuous, then T is an isomorphism of E with F. If in particular $E = F$, then T is an automorphism of E, and we denote $\mathrm{Aut}(E) \subset \mathcal{B}(E)$ the subspace of automorphisms. If in particular E is of finite dimension n, then $\mathrm{Aut}(E) = \mathrm{GL}(E)$.

Definition A.6. Topologies on $\mathcal{B}(E, F)$

Let $(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(E, F)$.

1. *We say that* $T_n \to T$ *in the* norm topology *if and only if* $\lim_{n\to\infty} \|T_n - T\| = 0$, *where* $\|\cdot\|$ *is the norm on* $\mathcal{B}(E, F)$.
2. *We say that* $T_n \to T$ *in the* strong operator topology *if and only if* $\lim_{n\to\infty} \|T_n x - Tx\|_F = 0$ *for all* $x \in E$.
3. *We say that* $T_n \to T$ *in the* weak operator topology *if* $\lim_{n\to\infty} \lambda(T_n x) = \lambda(Tx)$ *for all* $\lambda \in F^*$.

♣

In particular if E is a normed vector space over $k = \mathbb{R}$ or $k = \mathbb{C}$ and $F = k$, then $\mathcal{B}(E, k)$ is nothing but the dual E^* of E and the strong operator topology on $\mathcal{B}(E, k)$ is nothing but the weak-$*$-topology on E^*.

If $\mathcal{H}$ is a Hilbert space and $E = F = \mathcal{H}$, then the space of isometric isomorphisms of E $\mathrm{Iso}(E)$ is the space of unitary operators $\mathcal{U}(\mathcal{H})$. On $\mathcal{U}(\mathcal{H})$ the strong operator topology and the weak operator topology coincide.

Let G be a topological group and E a topological vector space. A *continuous representation of G on E* is a homomorphism $\pi\colon G \to \mathrm{Aut}(E)$, which is continuous with respect to a topology on $\mathrm{Aut}(E)$. If in particular, E is a normed space, then π is an *isometric representation* if $\pi\colon G \to \mathrm{Iso}(E)$. An isometric representation of a Hilbert space is called *unitary*.

Lemma A.3

Let G be a topological group acting continuously on a locally compact space X. Let $C_c(X)$ be the space of continuous functions with compact support on X with the norm topology. Then the representation $\pi\colon G \to \mathrm{Iso}(C_c(X))$ *defined by*

$$\pi(g)f(x) := f(g^{-1}x)$$

for $x \in X$ and $g \in G$ is a continuous representation if $\mathrm{Iso}(C_c(X))$ *is endowed with the strong operator topology.*

♡

If E, F are topological vector spaces and $T \in \mathcal{B}(E, F)$, the *adjoint* $T^* : F^* \to E^*$ is defined

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by

$$T^*(\lambda) := \lambda \circ T\,.$$

In particular, if E is a topological vector space on which G acts via a representation π, and E^* is endowed with the weak-$*$-topology, then

$$\pi^*(g) := \pi(g^{-1})^* \colon E^* \to E^*$$

is continuous.

Definition A.7. regular Borel measure

1. *Let X be a locally compact Hausdorff space. A measure on the σ-algebra of Borel sets of X is called a* Borel measure *if it is finite on every compact set.*
2. *A Borel measure μ is said to be* regular *if*
   - (a). *for every Borel set Y, $\mu(Y) = \sup \mu(K)$ over all compact subsets $K \subseteq Y$, and*
   - (b). *for every σ-bounded set Y, $\mu(Y) = \inf \mu(U)$ over all open σ-bounded sets $U \supseteq Y$ for every set U in B(X).*

♣

Recall that a set Y is σ-bounded if it is contained in the countable union of compact sets.

Definition A.8. Separability

Let $\mathcal{H}$ be a complex Hilbert space. We say that $\mathcal{H}$ is separable if it contains a countable dense subset. ♣

A.3 Differentiable Manifolds

Definition A.9. Paracompactness

A topological space X is paracompact if every open covering $\{U_\alpha\}_{\alpha\in A}$ has a locally finite refinement, that is there exists a covering $\{V_\beta\}_{\beta\in B}$ such that

- *For every $\beta \in B$ there exists at least one $\alpha \in A$ such that $V_\beta \subset U_\alpha$, and*
- *for every $p \in X$ there exists a neighborhood W of x that intersects finitely many V_β.* ♣

For us a *smooth manifold* will always be Hausdorff, locally Euclidean with countable basis and paracompact.

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Definition A.10. Germs

Given $p \in M$, we denote by $C^\infty(p)$ the algebra of germs *of smooth functions at p. This is the algebra of smooth functions defined in an open neighborhood of p, where two functions are identified if they coincide on a neighborhood of p.* ♣

Recall that the *tangent space* T_pM to the manifold M at the point p is the set of all linear functionals $X_p\colon C^\infty(p) \to \mathbb{R}$ such that for all $\alpha, \beta \in \mathbb{R}$ and all $f, g \in C^\infty(p)$:

1. $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$ (linearity);
2. $X_p(fg) = X_p(f) \cdot g(p) + f(p) X_p(g)$ (Leibniz rule).

The linear map $X_p \in T_pM$ is called a *tangent vector* to M at p and the tangent space T_pM has the structure of real vector space with operations:

1. $(X_p + Y_p)(f) := X_p(f) + Y_p(f)$;
2. $(\alpha X_p)(f) := \alpha X_p(f)$.

Let $f\colon M \to N$ be a smooth map of smooth manifolds and let $p \in M$. The *differential of f at p* is the linear map $d_pf\colon T_pM \to T_{f(p)}N$ defined as follows: if $X_p \in T_pM$ and $\phi \in C^\infty(f(p))$, then

$$d_pf(X_p) := X_p(\phi \circ f)\,.$$

In other words, the tangent vector $d_pf(X_p)$ applied to the function ϕ takes the derivative of the function $\phi \circ f$ at the point $p \in M$ in the direction of the tangent vector X_p.

The *tangent bundle* to M is $TM = \bigcup_{p \in M} T_pM$. It can be made into a manifold with coordinate charts $(U \times \mathbb{R}^n, \varphi \times \psi)$, where (U, φ) is a coordinate chart on M and $\psi\colon \mathbb{R}^n \to T_pM$ is an isomorphism. With this smooth structure the projection $\pi\colon TM \to M$ is smooth.

Definition A.11. Smooth vector field

A smooth vector field *is smooth section of the tangent bundle*

$$X\colon M \to TM$$

$\pi \circ X = id_M$. *In other words, it is a map*

$$\begin{aligned} X\colon M \to &\quad TM \\ p \mapsto &\; X_p \in T_pM \end{aligned}$$

-

that assigns to each point $p \in M$ a tangent vector X_p to M at p, and such that the map

$$\begin{aligned} Xf : M &\to \mathbb{R} \\ p &\mapsto X_p(f) \end{aligned}$$

is smooth, for every $f \in C^\infty(M)$. ♣

It can be proven that if $p \in M$, then

$$X_p(f) = d_p f(X_p)\,, \tag{A.1}$$

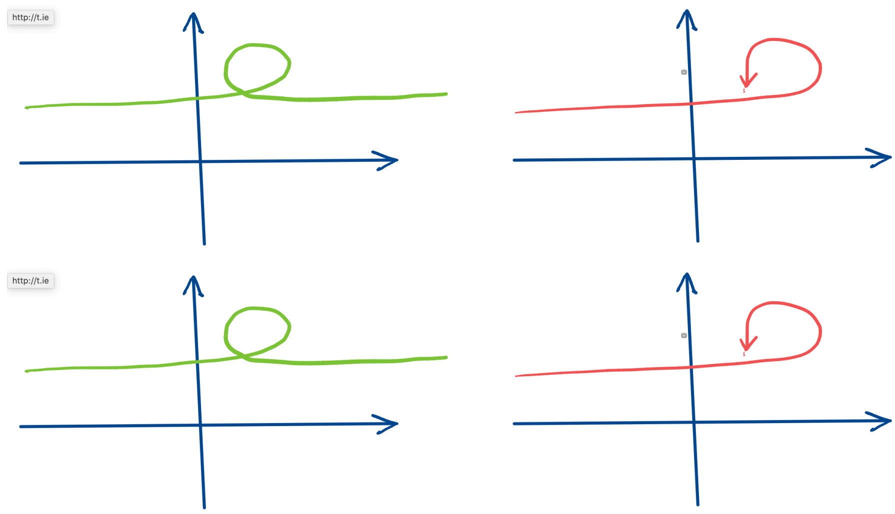
that is $X_p(f)$ is the differential of the function f at the point p in the direction of X_p.

Definition A.12

Let $\varphi : M \to N$ be a smooth map of smooth manifolds. Then:

1. *φ is an* immersion *if $d_p\varphi$ is non-singular for all $p \in M$.*
2. *$\varphi(M)$ is a* submanifold *or an* immersed sumbanifold *of N if φ is a one-to-one immersion.*
3. *If φ is a one-to-one immersion that is also a a homeomorphism of M onto its image, then φ is an embedding and $\varphi(M)$ is an embedded submanifold.* ♣

In the following pictures in green we see two immersion and in red two immersed submanifolds.



An embedded submanifold has the smooth structure coming from the ambient manifold and the concept of embedded submanifold are essentially equivalent to that of regular submanifold that

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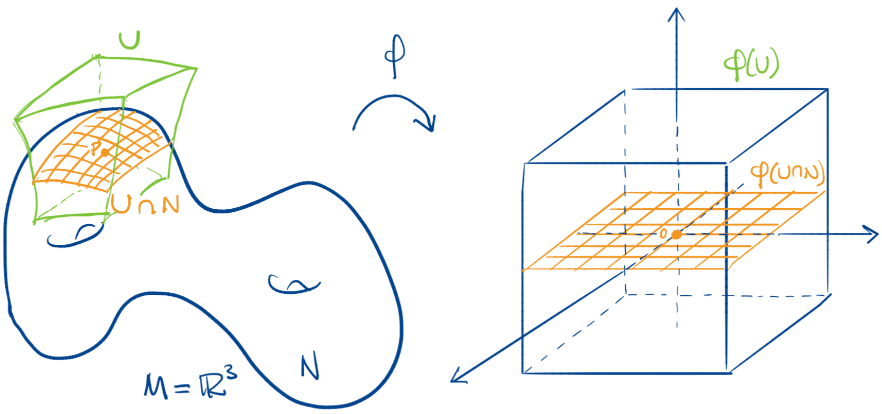
we recall now.

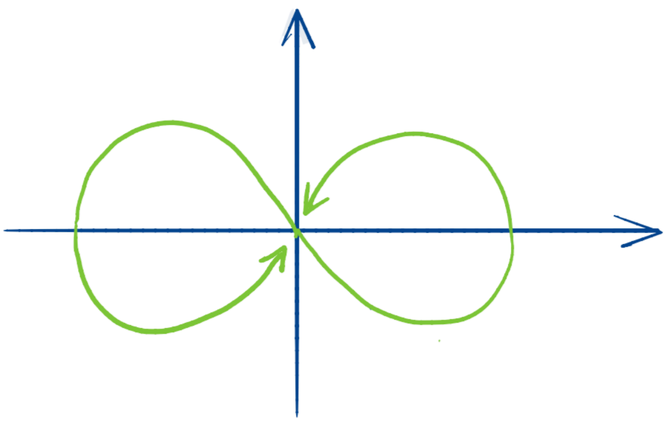
Definition A.13. (Regular Submanifold)

Let M be a smooth m-dimensional manifold.

1. *A subset $N \subset M$ has the* submanifold property *if every $p \in N$ has a coordinate neighborhood (U, φ) in M with local coordinates $x_1, \ldots, x_m$ such that*
   (a). $\varphi(p) = 0$;
   (b). $\varphi(U)$ *is an open cube* $(-\varepsilon, \varepsilon)^m$ *of side length* 2ε;
   (c). $\varphi(U \cap N) = \{x \in (-\varepsilon, \varepsilon)^m : x_{n+1} = \cdots x_m = 0\}$.
2. *A* regular submanifold *of M is any subset $N \subset M$ with the submanifold property and the smooth structure determined by the coordinate neighborhoods defined by the submanifold property.* ♣



Example A.2 The following is not a regular submanifold of $\mathbb{R}^2$.



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The point of a regular submanifold is that the topology and the differentiable structure are those derived from M.

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