Lecture

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Lie groups

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Algebraic approach / geometric approach

$G < GL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0\}$

Structure of the course

(1) Topological gps & Haar (m)

(2) Lie groups and their Lie algebras

(3) Structure theory: solvable, nilpotent, semisimple

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Today: Topological gps & examples

Tomorrow: which are compact or locally compact?

Why? Locally cpt gps have the Haar measure.

Defn. A topological gp G is a gp endowed with a Hausdorff topology w.r.t. which

$m: G\times G \to G$
$(g,h) \mapsto gh$

$i: G \to G$
$g \mapsto g^{-1}$

are continuous

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Recall A top. space is Hausdorff if any two pts have disjoint open nbds

Ex: The Zariski topology is not Hausdorff since closed sets are zeros of poly.

Operations on top. gps.

(1) $H < G$ sbgp, G top. gp $\Rightarrow$ $\Rightarrow$ H top. gp.

(2) Quotients of top. gps are top. gps.

(3) Products

(4) Semidirect products

Property If G_1, G_2 are top. gp. $p: G_1 \to G_2$ homo

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Then p cont. $\Longleftrightarrow$ p cont at one pt.

Ex1 Any gp with the discrete top.

Ex2 $(\mathbb{R}^n, +)$ is an Abelian top. gp w.r.t. the Euclidean topology

Ex3 $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$ are Abelian top. gps with the topology induced by the Euclidean topology.

Ex4 $M_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2} =$ $=$ $n\times n$ matrices with real coeff.

$GL(n,\mathbb{R}) := \{A \in \mathbb{R}^{n^2} : \det A \neq 0\}$

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is a top. gp with the top. induced by $\mathbb{R}^{n^2}$.

$GL(n,\mathbb{R})$ open in $\mathbb{R}^{n^2}$

- $A, B \in GL(n,\mathbb{R}) \Rightarrow$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

- $A \in GL(n,\mathbb{R}) \Rightarrow$

$$(A^{-1})_{ij} = \frac{\det M_{ij}}{\det A}$$

$M_{ij} = (i,j)$-minor

- $(A_k) \in GL(n,\mathbb{R})$, $A_k \to A$ $\Leftrightarrow$ $(A_k)_{ij} \to A_{ij}$

<u>Rk</u> Rather than $\mathbb{R}$, we can take any top. field k ($\mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, finite fields)

$\Rightarrow GL(n,k)$ is a top. gp.

<u>Ex.5</u> X cpt Hausdorff space

$\mathrm{Homeo}(X) := \{f : X \to X, \text{homeo}\}$ is a top. gp. with the compact-open topology

<u>Recall</u> If X, Y are top. spaces, $C(X,Y) := \{f : X \to Y \text{ cont.}\}$

The cpt-open top. is generated by the subbasis

$$S(C,U) := \{f \in C(X,Y) : f(C) \subset U\},$$ where

$C \subset X$ cpt, $U \subset Y$ open.

<u>Easy to see</u> If $(f_n) \in C(X,Y)$ then $f_n \to f$ in the cpt-open top $\Leftrightarrow$ $f_n \to f$ unif. on cpt sets.

If (Y,d) is a metric space

$$\Rightarrow B_C(f,\varepsilon) := \{g \in C(X,Y) : \sup_{x \in C} d(f(x), g(x)) < \varepsilon\},$$

where $C \subset X$ is cpt, $\varepsilon > 0$ is a basis for the cpt-op.

top. on $C(X,Y)$.

<u>Rk</u>

$$\left\{\begin{matrix}\text{unif.}\\ \text{conv.}\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}\text{unif.}\\ \text{conv.}\\ \text{on cpt}\\ \text{set}\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}\text{ptwise}\\ \text{conv.}\end{matrix}\right\}$$

$$\overset{=}{X \text{ cpt}} \qquad\qquad \overset{=}{X \text{ discrete}}$$

<u>Warning</u>: If X is not cpt.

$\Rightarrow \mathrm{Homeo}(X)$ can be very intractable

e.g. X loc. cpt $\Rightarrow$

$\Rightarrow \mathrm{Homeo}(X)$ need not be a top. ~~space~~ group

However:

- X loc. cpt & loc. conn $\Rightarrow$

  $\Rightarrow$ Homeo(X) top. gp.

- X proper metric space (closed balls of finite radius are cpt) $\Rightarrow$

  $\Rightarrow$ Homeo(X) top. gp.

<u>Ex 6</u> X cpt metric space

$$\mathrm{Iso}(X) := \{ f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x,y) \ \forall x, y \in X \}$$

is a top. gp.

<u>Ex 4</u> $GL(n,\mathbb{R}) < \mathrm{Homeo}(\mathbb{R}^n)$ with the topology of unif conv. on cpt sets.

$(A_k) \subset GL(n,\mathbb{R})$, $A_k \to A$ unif on cpt set $\Rightarrow$

$\Rightarrow$ A linear and $GL(n,\mathbb{R})$ is also a top. gp. w.r.t. the cpt-open top.

<u>Exercise</u> The cpt-open top. and the Euclidean top. on $GL(n,\mathbb{R})$ coincide

<u>Ex. 7</u> M differ. mfd

$$\mathrm{Diff}^r(M) := \{ f : M \to M, \ f, f^{-1} \in C^r(M) \}$$

is a top. gp but if $(f_n) \in \mathrm{Diff}^r(M)$, $f_n \to f$ in the cpt. open top $\Rightarrow$

$f \notin \mathrm{Diff}^r(M)$.

Instead $f_n \to f \Leftrightarrow$

$\Leftrightarrow$ $f_n^{(k)} \to f^{(k)}$

$(f_n^{-1})^{(k)} \to (f^{-1})^{(k)}$

unif on cpt-sets $\forall k \le r$.

<u>Ex. 8</u> Λ partially ordered set $(G_\lambda)_{\lambda \in \Lambda}$ family of gps s.t.

$\forall \lambda_1, \lambda_2$ with $\lambda_1 \le \lambda_2$ $\exists$ homo $G_{\lambda_2} \xrightarrow{p_{\lambda_2, \lambda_1}} G_{\lambda_1}$.

$((G_\lambda)_{\lambda \in \Lambda}, p_{\lambda_2, \lambda_1})$ is a <u>projective system</u> if

- $p_{\lambda, \lambda} = \mathrm{id}_{G_\lambda}$
- if $\lambda_1 \le \lambda_2 \le \lambda_3$ then

  $p_{\lambda_3, \lambda_1} = p_{\lambda_3, \lambda_2} \circ p_{\lambda_2, \lambda_1}$.

The inverse limit of this projective system is the unique smallest top. gp G s.t.

s.t. $\forall \lambda \in \Lambda$ $\exists$ homo

$p_\lambda : G \to G_\lambda$ s.t.

$$\begin{tikzcd} G \arrow[r, "p_{\lambda_2}"] \arrow[dr, "p_{\lambda_1}"'] & G_{\lambda_2} \arrow[d, "p_{\lambda_2,\lambda_1}"] \\ & G_{\lambda_1} \end{tikzcd}$$

<u>Notation</u>

$$\varprojlim G_\lambda := \{(x_\lambda)_{\lambda\in\Lambda} \in \prod G_\lambda : p_{\lambda_2,\lambda_1}(x_{\lambda_2}) = x_{\lambda_1}\}$$

= set of compatible sequences

<u>Fact</u> $G = \varprojlim G_\lambda$

- (G_λ) top. gps $\Rightarrow$ $\prod G_\lambda$ top. gp & $\varprojlim G_\lambda$ top. gp.
- (G_λ) cpt $\Rightarrow$ $\varprojlim G_\lambda$ cpt
- (G_λ) discrete $\Rightarrow$ $\varprojlim G_\lambda$ is totally disc.

<u>Defn</u> (G_λ) finite gps $\Rightarrow$

$\Rightarrow$ $\varprojlim G_\lambda$ is called a <u>profinite</u> gp

<u>Ex.</u> $\Lambda = \mathbb{N}$

$$\left((\mathbb{Z}/p^n\mathbb{Z})_{n\in\mathbb{Z}},\ p_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \right)$$

reduction mod p^m

<u>Fact</u> The inverse limit of this projective systems are the p-adic integers $\mathbb{Z}_p$. They are cpt and in fact a compactifica of $\mathbb{Z}$.

__Ex 8__ Important subgps of $GL(n,\mathbb{R})$.

(a) $A_{det} = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} ; \lambda_i \neq 0 \right\}$

$A_{det} \cong (\mathbb{R}^*)^n$ Abel. top. gp.

$A := A_1$

(b) $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL(n,\mathbb{R}) \right\}$

is a closed subgp. of $GL(n,\mathbb{R})$ homeom. to $\mathbb{R}^{n\cancel{(n+1)}\,(n-1)/2}$ (as top. space but not gps if $n \geq 3$). For ex. N is not Abelian if $n \geq 3$.

(c) $K := \{ X \in GL(n,\mathbb{R}) : \forall v, w \in \mathbb{R}^n$
$\langle Xv, Xw \rangle = \langle v, w \rangle \} =$

$= \{ X \in GL(n,\mathbb{R}) : \forall v \in \mathbb{R}^n$
$\|Xv\| = \|v\| \}$

$= \{ X \in GL(n,\mathbb{R}) : {}^tX X = Id \} =$

$=: O(n,\mathbb{R})$

More generally, if V is a __real__ vector space and $B : V \times V \to \mathbb{R}$ non-deg. bilinear form on V

$O(V,B) := \{ A \in GL(V) ;$
$\forall v, w \in V,$
$B(Av, Aw) = B(v,w) \}$

$O(V,B)$ is the orthogonal gp of B. $\Rightarrow$ $O(n,\mathbb{R})$ is the orthogonal gp of the usual inner product on $\mathbb{R}^n$.

We can choose a basis of V s.t.

$$B_p(v,w) = -\sum_{j=1}^{p} v_j w_j + \sum_{j=p+1}^{n} v_j w_j$$

$B_p \gg \Leftrightarrow p = 0$, in which case $B_0 = \langle \, , \rangle$

$O(V, B_p) =: O(p,q)$, where $p+q = n$.

If V is a $\mathbb{C}$-vector space $\Rightarrow (e_1, \dots, e_p, e_{p+1}, \dots, e_n) \rightsquigarrow$

$\rightsquigarrow (ie_1, \dots, ie_p, e_{p+1}, \dots, e_n)$ and hence $B_p = \langle \, , \rangle$

__Ex. 9__ V complex v.s.

$h : V \times V \to \mathbb{C}$ Hermitian inner product, i.e. pos. defn, antisymm., complex valued, linear in the first var & antilinear in the second

$U(V,h) =$ unitary gp of (V,h)

$= \{ X \in GL(V) : h(Xv, Xw) =$
$= h(v,w), \ v, w \in V \}$

$= \{ X \in GL(V) : X^* = X^{-1} \}$

$X^* = \overline{X}^t$

If $x \in U(V,h)$ then
$|\det x| = 1$

For example $h_0 : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$
$$h_0(x,y) = \sum_{j=1}^{n} x_j \bar{y}_j$$
and we write
$$U(n) := U(\mathbb{C}^n, h_0)$$

Recap on topologies on operator spaces

E, F normed space
$$B(E,F) := \{T : E \to F \text{ cont. \& linear}\}$$
is also normed with
$$\|T\| = \sup_{\|x\|_E = 1} \|Tx\|_F$$

Rk If T are 1-1 with cont. inverse $\Rightarrow$
$\Rightarrow B(E,E) \overset{?}{\supseteq} \mathrm{Aut}(E)$

If $\dim E < \infty \Rightarrow$
$\Rightarrow \mathrm{Aut}(E) = GL(E)$

Topologies on $B(E,F)$

$(T_n) \in B(E,F)$

- $T_n \to T$ in the norm topology $\Leftrightarrow \|T_n - T\| \xrightarrow[n\to\infty]{} 0$
- $T_n \to T$ in the strong operator topology $\Leftrightarrow$ $\|T_n x - Tx\|_F \xrightarrow[n\to\infty]{} 0$ $\forall x \in E$
- $T_n \to T$ in the weak operator topology $\Leftrightarrow$
$$\lambda(T_n x) \to \lambda(Tx)$$
$\forall x \in E$ and $\forall \lambda \in F^*$

Rk (1) If $F = \mathbb{k}$ and E is normed over $\mathbb{k}$ $\Rightarrow$
$\Rightarrow B(E,\mathbb{k}) = E^*$ and the strong op. top. on $B(E,\mathbb{k})$ is the weak-$*$-top. on E^*.

(2) If $E = F = \mathcal{H}$ is a separable Hilbert space $\Rightarrow \mathrm{Iso}(\mathcal{H}) = \mathcal{U}(\mathcal{H}) =$
$= \{U \in B(\mathcal{H},\mathcal{H}) : U^* U = \mathrm{Id}\}$
and here the strong op. top. = weak op. top.

Ex. 9 $\mathcal{U}(\mathcal{H})$ is a top. gp w.r.t. the strong operator topology (or weak op. top.)

Compact & local comp.

Recall If X is a l.c. top. gp. and $Y \in X$ is open or closed $\Rightarrow Y$ is l.c.t.g.

Ex Discrete top. gp. are l.c.

Ex. $(\mathbb{R}^n,+)$, $(\mathbb{R}^*,\cdot)$, $(\mathbb{C}^*,\cdot)$ are l.c.

Ex $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$ are l.c.

Ex. X cpt $\Rightarrow$ Homeo(X) top. gp with the cpt-open top. that is not nec. locally cpt.

Exercise Homeo(S^1) is not locally compact

Ex X metric space $\Rightarrow$ Iso(X) is "as good as X".

- X cpt $\Rightarrow$ Iso(X) cpt
- X l.c. $\Rightarrow$ Iso(X) l.c.

Ascoli-Arzelà (X,d_X), (Y,d_Y) cpt. metric space

$C(X,Y)$ with

$$d(f,g) := \sup_{x\in X} d_Y(f(x),g(x))$$

Then a family $\mathcal{F}\subset C(X,Y)$ is rel. cpt $\Leftrightarrow$ $\mathcal{F}$ is equicont., that is $\forall\varepsilon>0$ $\exists\,\delta>0$ s.t. $d_Y(f(x),\cancel{g}\,f(y))<\varepsilon$

$\forall\, x,y\in X$ s.t. $d_X(x,y)<\delta$ and forall $f\in\mathcal{F}$

Rk
- X need not be metric
- Y need not be cpt if $\{f(x) : x\in X, f\in\mathcal{F}\}$ is rel. cpt.

For us $X=Y$, $\mathcal{F}=$ Iso(X)

$\Rightarrow$ $\underbrace{\text{Iso}(X)}_{\text{much much smaller than}}\subset$ Homeo$(X)\subset C(X,X)$