EXERCISE SHEET 1

(1) Let $\mathfrak{gl}_n$ be the algebra of $n \times n$ matrices over complex numbers $\mathbb{C}$. Define a norm on $\mathfrak{gl}_n$,

$$\begin{aligned} \|\cdot\|\colon \mathfrak{gl}_n &\to \mathbb{R}_{\geq 0} \\ A = (a_{i,j})_{i,j} &\mapsto \sqrt{\sum |a_{i,j}|^2}, \end{aligned}$$

where $|a_{i,j}|^2 := a_{i,j} \cdot \overline{a}_{i,j}$ for a complex number $a_{i,j}$ and its conjugate $\overline{a}_{i,j}$.

(1.1) Show that $\|\cdot\|$ is *submultiplicative*, i.e., for a pair of matrices A and B we have

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|.$$

(1.2) Deduce that the series

$$\sum_{k \geq 0} \frac{A^k}{k!} = I_n + A + \frac{A^2}{2} + \dots$$

converges in $\mathfrak{gl}_n$ with respect to the norm $\|\cdot\|$. The limit is the *exponential* of A,

$$e^A := \sum_{k \geq 0} \frac{A^k}{k!}.$$

(2) Given a matrix A. Show that

$$\det(e^A) = e^{\operatorname{tr}(A)}.$$

Conclude that e^A is always invertible. (Hint: try a diagonal matrix first.)

(3) (3.1) Show that if $[A, B] = A \cdot B - B \cdot A = 0$, then

$$e^A \cdot e^B = e^{A+B}.$$

(3.2) Find a pair of matrices A and B, such that

$$e^A \cdot e^B \neq e^{A+B}.$$

(4) Given Lie algebras $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{a}$, such that $\mathfrak{a}$ is abelian. We say that $\mathfrak{h}$ is a central extension of $\mathfrak{g}$ by $\mathfrak{a}$, if

- there exists a sequence of Lie-algebra homomorphisms

$$\mathfrak{a} \xhookrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g},$$

such that $\ker(\pi) = \operatorname{Im}(\iota)$;
- elements of $\mathfrak{a}$ commute with elements of $\mathfrak{h}$, i.e., $[\mathfrak{a}, \mathfrak{h}]_{\mathfrak{h}} = 0$ (identifying $\mathfrak{a}$ with $\operatorname{Im}(\iota)$).

Two central extensions $\mathfrak{h}$, $\mathfrak{h}'$ of $\mathfrak{g}$ by $\mathfrak{a}$ are equivalent, if there exists a Lie-algebra isomorphism between $\mathfrak{h}$ and $\mathfrak{h}'$ which commutes with π and ι,

$$\begin{array}{ccccc} \mathfrak{a} & \xhookrightarrow{\iota} & \mathfrak{h} & \xrightarrow{\pi} & \mathfrak{g} \\ \big\downarrow{=} & & \big\downarrow{\cong} & & \big\downarrow{=} \\ \mathfrak{a} & \xhookrightarrow{\iota'} & \mathfrak{h}' & \xrightarrow{\pi'} & \mathfrak{g} \end{array}$$

(4.1) Let $\beta\colon \mathfrak{g} \to \mathfrak{h}$ be a linear map (not necessarily a Lie-algebra homomorphism), such that $\pi \circ \beta = \mathrm{id}_{\mathfrak{g}}$. Define

$$\Theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h},$$
$$\Theta(x, y) := [\beta(x), \beta(y)] - \beta([x, y]).$$

Show that Θ is skew-symmetric, $\Theta(x, y) \in \mathfrak{a}$, and it satisfies the following property:

$$(\star) \qquad \Theta(x, [y, z]) + \Theta(y, [z, x]) + \Theta(z, [x, y]) = 0.$$

(4.2) Conversely, given a skew-symmetric bilinear map $\Theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ which satisfies the property $(\star)$. Show that

$$[(x, z), (y, z')]_\Theta := ([x, y]_{\mathfrak{g}}, \Theta(x, y)), \quad \text{for } (x, z), (y, z') \in \mathfrak{g} \oplus \mathfrak{a},$$

is a Lie bracket on the vector space $\mathfrak{g} \oplus \mathfrak{a}$, such that $(\mathfrak{g} \oplus \mathfrak{a}, [,]_\Theta)$ is a central extension of $\mathfrak{g}$ by $\mathfrak{a}$. We call such Θ a *cocycle*.

(4.3) Show that for a cocycle Θ associated to a central extension $\mathfrak{h}$ via the construction from (4.1), there is an equivalence of central extensions,

$$(\mathfrak{h}, [,]_{\mathfrak{h}}) \cong (\mathfrak{g} \oplus \mathfrak{a}, [,]_\Theta).$$

(4.4) Consider

$$\Theta_\mu(x, y) := \mu([x, y])$$

for some linear map $\mu\colon \mathfrak{g} \to \mathfrak{a}$. Show that Θ_μ is a cocycle, and that there exists a Lie-algebra isomorphism

$$(\mathfrak{g} \oplus \mathfrak{a}, [,]_{\Theta_\mu}) \cong (\mathfrak{g} \oplus \mathfrak{a}, [,]),$$

where $(\mathfrak{g} \oplus \mathfrak{a}, [,])$ is the direct sum of Lie algebras (i.e., the trivial central extension). More generally, show that for any cocycle Θ and a linear map μ, there exists a Lie-algebra isomorphism

$$(\mathfrak{g} \oplus \mathfrak{a}, [,]_{\Theta+\Theta_\mu}) \cong (\mathfrak{g} \oplus \mathfrak{a}, [,]_\Theta).$$

(4.5) Consider the vector spaces of all cocycles Θ and cocycles of the form Θ_μ for a linear map $\mu\colon \mathfrak{g} \to \mathfrak{a}$,

$$Z^2(\mathfrak{g}, \mathfrak{a}) = \{ \text{ cocycles } \Theta \ \},$$
$$B^2(\mathfrak{g}, \mathfrak{a}) = \{ \text{ cocycles } \Theta_\mu \}.$$

Conclude that the equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$ are in one-to-one correspondence with the quotient

$$H^2(\mathfrak{g}, \mathfrak{a}) = Z^2(\mathfrak{g}, \mathfrak{a}) / B^2(\mathfrak{g}, \mathfrak{a}).$$