EXERCISE SHEET 2

(1) Schur's lemma states that over an algebraically closed field every endomorphism of a finite-dimensional irreducible representation of a Lie algebra is a scalar multiple of the identity (i.e., it is given by multiplication with a scalar). The proof of Schur's lemma relies on the existence of eigenvalues of matrices. Eigenvalues might not exist in the infinite-dimensional case, hence the same proof cannot be applied.

We will now establish an infinite-dimensional version of Schur's lemma, called Dixmier's lemma (it specialises to Schur's lemma over $\mathbb{C}$).

Lemma (Dixmier's lemma)**.** *Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let V be an irreducible $\mathfrak{g}$-representation of countable dimension. Then every endomorphism of V is a scalar multiple of the identity.*

Proof.

(1.1) Let $\mathrm{End}_{\mathfrak{g}}(V)$ be the algebra of endomorphisms of the representation V. Show that $\mathrm{End}_{\mathfrak{g}}(V)$ is a division algebra, i.e., every non-zero element of $\mathrm{End}_{\mathfrak{g}}(V)$ has a multiplicative inverse.

(1.2) Given an element $\phi \in \mathrm{End}_{\mathfrak{g}}(V)$. Show that for all non-zero $v \in V$, ϕ is completely determined by $\phi(v)$. That is for all non-zero $v \in V$, the following holds: if $\phi(v) = \phi'(v)$, then $\phi = \phi'$. (Hint: use the irreducibility of V.)

(1.3) Deduce that there exists an injective linear map

$$\mathrm{End}_{\mathfrak{g}}(V) \hookrightarrow V,$$

and, in particular, that $\mathrm{End}_{\mathfrak{g}}(V)$ is of countable dimension.

(1.4) Show that every division algebra D of countable dimension over $\mathbb{C}$ must be $\mathbb{C}$ itself. (Hint: assuming the contrary, argue that D must contain a transcendental element which can be used to deduce a contradiction to the countability of dimension; uncountability of $\mathbb{C}$ must be used at some point.)

(1.5) Conclude that Dixmier's lemma holds. Observe that the assumption that the Lie algebra is defined over $\mathbb{C}$ is necessary, as the same proof does not apply to a countable algebraically closed field.

□

(2) Let $\mathfrak{g}$ be a Lie algebra of countable dimension over $\mathbb{C}$. We call an element $c \in \mathfrak{g}$ central if

$$[c, x] = 0, \text{ for all } x \in \mathfrak{g}.$$

Show that a central element c acts by a multiple of the identity on all irreducible representations of $\mathfrak{g}$.

(3) Let $\mathfrak{Heis} = \{a_n, \mathbb{1} \mid n \in \mathbb{Z}\}$ be the infinite-dimensional Heisenberg algebra. Let $B = \mathbb{C}[x_1, x_2, \ldots]$ be the Bosonic Fock space of $\mathfrak{Heis}$. Consider a representation V of $\mathfrak{Heis}$ with a non-zero vector $v \in V$ which satisfies the following properties:

$$\begin{aligned} &a_n \cdot v = 0 \quad \text{for } n > 0 \\ &a_0 \cdot v = \mu v \quad \text{for } \mu \in \mathbb{C} \\ &\mathbb{1} \cdot v = v. \end{aligned}$$

Show that the linear map

$$\phi\colon B \to V, \quad p(x_1, \ldots, x_n) \mapsto p(a_{-1}, \ldots, \frac{a_{-n}}{n}) \cdot v$$

is a homomorphism of representations of $\mathfrak{Heis}$.

(4) Recall an involution ω on $\mathfrak{Heis}$,

$$\omega\colon \mathfrak{Heis} \xrightarrow{\sim} \mathfrak{Heis}, \quad \omega(\lambda a_n) = \overline{\lambda} a_{-n},\ \omega(\lambda \mathbb{1}) = \overline{\lambda}\mathbb{1}.$$

Define a pairing on the Bosonic Fock space B,

$$\langle p \mid q \rangle := \langle \omega(p) \cdot q \rangle, \quad p, q \in B,$$

where $\langle\ \rangle$ is the constant term of a polynomial. Show that $\langle\ \mid\ \rangle$ is a non-degenerate Hermitian pairing, such that monomials $x_1^{k_1} \ldots x_n^{k_n}$ form an orthogonal basis with

$$\langle 1 \mid 1 \rangle = 1, \quad \langle x_1^{k_1} \ldots x_n^{k_n} \mid x_1^{k_1} \ldots x_n^{k_n} \rangle = \prod_{j=1}^{n} k_j! j^{k_j}.$$