EXERCISE SHEET 3

(1) Let $\mathfrak{Vir} = \{d_n, C \mid n \in \mathbb{Z}\}$ be the Virasoro algebra, and V be a representation of $\mathfrak{Vir}$, which as a vector space decomposes into the direct sum of eigenspaces of d_0,

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda, \tag{1}$$

where $V_\lambda = \{v \in V \mid d_0 \cdot v = \lambda v\}$. Let $V' \subseteq V$ be a subrepresentation, and consider an expression of a vector $v \in V'$ in terms of w_i,

$$v = \sum_{i=1}^{m} w_i,$$

such that $d_0 \cdot w_i = \lambda_i w_i$ for some λ_i, which is possible by (1). Show that

$$w_i = \sum_{j=0}^{m-1} a_j d_0^j \cdot v$$

for some $a_j \in \mathbb{C}$. Conclude that V' respects the eigenspace decomposition,

$$V' = \bigoplus_{\lambda \in \mathbb{C}} (V' \cap V_\lambda).$$

(Hint: there is nothing special about $\mathfrak{Vir}$ in this exercise, use only d_0; recall the Vandermonde matrix.)

(2) A highest weight representation of $\mathfrak{Vir}$ is a representation V, which admits a non-zero vector $v \in V$, such that:

$$C \cdot v = cv$$
$$d_0 \cdot v = hv,$$

for some complex numbers c and h, and V is spanned by vectors of the form

$$d_{-i_k} \dots d_{-i_1} \cdot v, \quad \text{for } 0 < i_1 \le \dots \le i_k,$$

we call v the highest weight vector, while (c, h) the highest weight (recall the same terminology for $\mathfrak{sl}_2$ from the lecture notes).

Show that a highest weight representation of V admits a vector-space decomposition

$$V = \bigoplus_{j \in \mathbb{Z}_{>0}} V_{h+j},$$

where $V_{h+j} = \{w \in V \mid d_0 \cdot w = (h + j)w\}$. Conclude, in particular, that $d_i \cdot v = 0$ for $i > 0$.

(3) Using Exercises (1) and (2), prove that a highest weight representation of $\mathfrak{Vir}$ is irreducible, if and only if the multiples of the highest weight vector v are the only non-zero vectors which satisfy

$$d_i \cdot v = 0, \qquad \text{for } i > 0.$$

We call a vector with the property above singular.

(4) Recall from the notes the representation of $\mathfrak{Vir}$ on $B = \mathbb{C}[x_1, x_2, \ldots]$, induced by the operators

$$L_k = \frac{1}{2}\sum_j : a_{-j}a_{j+k} : \; = \frac{\epsilon}{2}a_{k/2}^2 + \sum_{j>-k/2} a_{-j}a_{j+k},$$

where ϵ is 0, if k is odd, and is 1, if k is even; a_j are Heisenberg creation and annihilation operators, such that a_0 acts by multiplication with a complex number μ. For this exercise, we make the substitution

$$a_n \mapsto a_n' = \sqrt{2}a_n, \quad n > 0$$
$$a_{-n} \mapsto a_{-n}' = \frac{1}{\sqrt{2}}a_{-n}, \quad n \geq 0,$$

to obtain operators $L_k' = \frac{1}{2}\sum_j : a_{-j}'a_{j+k}' :$.

(4.1) Let a_0' act on B by $\mu = 0$. Show that the polynomial x_1 is a singular vector, that is

$$L_k' \cdot x_1 = 0, \qquad \text{for all } k > 0.$$

(4.2) Let a_0' act by $\mu = \frac{-1}{\sqrt{2}}$. Show that $x_1^2/2 + x_2$ is a singular vector.

(4.3) More generally, show that if a_0' acts by $\mu = \frac{-m}{\sqrt{2}}$, $m \in \mathbb{Z}_{>0}$, then the polynomial

$$S_{m+1} = \sum_{k_1+2k_2+\ldots=m+1} \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k_2}}{k_2!}\cdots$$

is singular.

Using Exercise (3), conclude that B is not an irreducible representation of $\mathfrak{Vir}$ for values of μ as above.

Polynomials S_m are examples of Schur polynomials, which will play a very important role later in the course. In fact, the observation above can be generalized to a statement that completely characterises singular vectors in B. There exist singular vectors distinct from 1 only when μ is an integer, in which case they are given by certain Schur polynomials.