EXERCISE SHEET 4

Let

$$V = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} v_i$$

be a vector space of countable dimension with a fixed basis given by vectors v_i, $i \in \mathbb{Z}$. Recall from the lectures the Infinite wedge,

$$F = \wedge^{\infty} V = \bigoplus_{m \in \mathbb{Z}} F^{(m)},$$

where the space $F^{(m)}$ is spanned by semi-infinite monomials,

$$\psi = v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \dots,$$

such that $s_i \in \mathbb{Z}$ are subject to the following conditions:

- $s_0 > s_{-1} > s_{-2} > \dots$
- $s_i = i + m$ for $i \ll 0$.

Each ψ is determined by the set of indices of vectors v_i appearing in ψ,

$$S = \{s_0, s_{-1}, s_{-2}, \dots\} \subset \mathbb{Z},$$

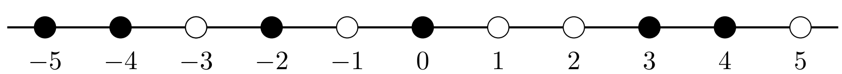
which can be given a pictorial representation in terms of Maya diagrams (see lecture notes for how to associate a Maya diagram to a set S).

(1) Consider a semi-infinite monomial

$$\psi = v_6 \wedge v_5 \wedge v_4 \wedge v_3 \wedge v_{-1} \wedge v_{-2} \wedge v_{-3} \wedge \dots,$$

such that v_i stabilises after dots. Draw the corresponding Maya diagram. In what subspace $F^{(m)}$ the semi-infinite monomial ψ lives?

(2) Consider a Maya diagram



What is the corresponding semi-infinite monomial and in what subspace $F^{(m)}$ it lives?

(3) Write as many semi-infinite monomials as you can and draw the corresponding Maya diagrams. Vice versa, draw as many Maya diagrams as you can and write the corresponding semi-infinite monomials.

(4) To each set $S = \{s_0, s_{-1}, s_{-2}, \dots\}$ associated to a semi-infinite monomial in $F^{(m)}$, we can associate a partition $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_\ell)$,

$$\lambda_i = s_{-i} + i - m, \quad i \in \mathbb{Z}_{\geq 0},$$

such that all zero λ_i are disregarded.

(4.1) Verify that λ is indeed a partition, i.e., it is a finite set of positive integers, which moreover satisfies $\lambda_i \geq \lambda_{i+1}$.
(4.2) What are the partitions associated to semi-infinite monomials considered in Exercises (1) and (2)?
(4.3) Show that sets S associated to a subspace $F^{(m)}$ for a fixed m and partitions λ are in one-to-one correspondence via the prescription $\lambda_i = s_{-i} + i - m$ (we allow empty partitions).
(4.4) For simplicity, let $m = 0$, verify the equality

$$\sum_{s \in S_+} s - \sum_{s \in S_-} s = |\lambda| = \sum_i \lambda_i,$$

where λ is a partition associated to a set S, and $S_+ = \{s \in S \mid s > 0\}$, $S_- = \{s \notin S \mid s \leq 0\}$.

(5) Convince yourself that Maya diagrams, through the interpretation of Dirac's sea of electrons, provide a metaphysical evidence for the existence of the antiparticle "positron".