EXERCISE SHEET 5

(1) Consider a subspace $F^{(m)}$ of the Infinite wedge $\wedge^\infty V$. We define a Hermitian form $\langle \mid \rangle$ on $F^{(m)}$ by declaring semi-infinite monomials to be orthonormal. More precisely, let ψ_S and $\psi_{S'}$ be two semi-infinite monomials associated to sets S and S' in $F^{(m)}$, then

$$\langle \psi_S \mid \psi_{S'} \rangle = \delta_{S,S'}.$$

Show that the representation $\rho\colon \mathfrak{gl}_\infty \to \mathfrak{gl}(F^{(m)})$ defined in the lecture notes satisfies

$$\langle \rho(M)\psi_S \mid \psi_{S'} \rangle = \langle \psi_S \mid \rho(M^\dagger)\psi_{S'} \rangle,$$

where $M^\dagger$ is the transpose complex conjugate of the matrix M, i.e., $M^\dagger_{ij} = \overline{M}_{ji}$. We call such representations unitary.[1]

(2) (2.1) Using the Hermitian form $\langle \mid \rangle$, show that $F^{(m)}$ splits as a direct sum of its irreducible subrepresentations (Hint: use orthogonal complements).

(2.2) Show that by acting on $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots$ with matrices in $\mathfrak{gl}_\infty$, we can "create" all other semi-infinite monomials.

(2.3) Deduce that $F^{(m)}$ must be an irreducible representation of $\mathfrak{gl}_\infty$.

(3) Recall operators $d_n \in \overline{\mathfrak{a}}_\infty$ from the lecture notes, defined by

$$d_n v_k = k v_{k-n},$$

whose expression in terms of elementary matrices is $d_n = \sum_i (i+n) E_{i,i+n}$. Compute the commutation relation of two operators d_n inside the central extension $\mathfrak{a}_\infty$,

$$[d_n, d_m]_{\mathfrak{a}_\infty} = ?$$

More generally, for two complex numbers α and β, consider operators $d_n^{\alpha,\beta} \in \overline{\mathfrak{a}}_\infty$ defined by

$$d_n^{\alpha,\beta} v_k = (k - \alpha - \beta(n+1)) v_{k-n}.$$

Compute their commutation relation inside $\mathfrak{a}_\infty$,

$$[d_n^{\alpha,\beta}, d_m^{\alpha,\beta}]_{\mathfrak{a}_\infty} = ?$$

An action of what algebra do these operators induce on $F^{(m)}$?

[1] We have already seen a representation with a similar property, namely $\mathfrak{Heis} \curvearrowright B$, where instead of the transpose conjugate we considered the involution $\omega(a_n) = a_{-n}$. More generally, unitary representations are those that satisfy the property stated in the exercise with respect to some involution ω on a Lie algebra $\mathfrak{g}$.