EXERCISE SHEET 6

(1) Show that the Hermitian form $\langle \mid \rangle$ on B defined in Section 3.2.3 of lecture notes is uniquely determined by the following two properties,

$$\langle 1 \mid 1 \rangle = 1, \quad \langle a_n \cdot p \mid q \rangle = \langle p \mid a_{-n} \cdot q \rangle.$$

Using Exercise (1) of Exercise sheet 5, deduce that the Boson–Fermion isomorphism,

$$\sigma \colon F^{(m)} \xrightarrow{\sim} B,$$

respects the Hermitian forms.

(2) Recall from the lecture notes the contracting and wedging operators, $\check{v}_i^*$ and $\hat{v}_i$, acting on the infinite wedge $\wedge^\infty V$.

(2.1) What are their degrees with respect to the decomposition $\wedge^\infty V = \bigoplus_m F^{(m)}$ (i.e., where do elements of $F^{(m)}$ are sent to)?

(2.2) Show that they satisfy the following (anti)-commutation relations:

$$[\hat{v}_i, \hat{v}_j]_+ := \hat{v}_i \hat{v}_j + \hat{v}_j \hat{v}_i = 0, \quad [\check{v}_i^*, \check{v}_j^*]_+ = 0, \quad [\hat{v}_i, \check{v}_j^*]_+ = \delta_{ij}.$$

(3) Show that the modified operators $\hat{\rho}(E_{ij})$ acting on $\wedge^\infty V$ defined in the lectures notes can be expressed as follows in terms of contracting and wedging operators,

$$\hat{\rho}(E_{ij}) = \begin{cases} \hat{v}_i \check{v}_j^* & \text{if } j > 0 \\ -\check{v}_j^* \hat{v}_i & \text{if } j \leq 0. \end{cases}$$