EXERCISE SHEET 7

(1) Consider the vector space $V = \bigoplus \mathbb{C} v_i$. Let $v_i^* \in \mathrm{Hom}(V, \mathbb{C})$ be defined by

$$v_i^*(v_j) = \delta_{ij}.$$

Show that the following definition of wedging and contracting operators agree with the definition from the lecture notes,

$$\begin{aligned}
\check{v}_i^*(v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \dots) &:= v_i^*(v_{s_0}) \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \dots \\
&- v_{s_0} \wedge v_i^*(v_{s_{-1}}) \wedge v_{s_{-2}} \wedge \dots \\
&+ v_{s_0} \wedge v_{s_{-1}} \wedge v_i^*(v_{s_{-2}}) \wedge \dots \\
&- \dots \\
\hat{v}_i(v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \dots) &:= v_i \wedge v_{s_0} \wedge v_{s_0} \wedge v_{s_{-2}} \wedge \dots
\end{aligned}$$

(2) Show that $\check{v}_i^*$ and $\hat{v}_i$ are adjoint with respect to the Hermitian pairing $\langle \mid \rangle$ on $\wedge^\infty V$, defined in Exercise sheet 5, that is,

$$\langle \hat{v}_i \psi \mid \psi' \rangle = \langle \psi \mid \check{v}_i^* \psi' \rangle.$$

(3) (Wick's Theorem, taken from Séverin Charbonnier) For $i = 1, \dots k$, let

$$\Psi_{n_i} = \sum_{n \geq 0} a_{i,n} \hat{v}_{n_i - n}, \quad \Psi_{m_i}^* = \sum_{m \geq 0} b_{i,m} \check{v}_{m_i + m}^*,$$

for some complex numbers $a_{i,n}$ and $b_{i,n}$.

(3.1) Show that $\langle \psi_0 \mid \Psi_{n_i} \Psi_{m_j}^* \psi_0 \rangle$ is well-defined.

(3.2) By induction, show Wick's theorem,

$$\langle \psi_0 \mid \Psi_{n_1} \dots \Psi_{n_k} \Psi_{m_k}^* \dots \Psi_{m_1}^* \psi_0 \rangle = \det_{i,j=1,\dots,k} (\langle \psi_0 \mid \Psi_{n_j} \Psi_{m_i}^* \psi_0 \rangle).$$

(4) Recall from the lectures the group GL_∞,

$$\mathrm{GL}_\infty = \left\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \;\middle|\; \begin{array}{c} A \text{ is invertible,} \\ \text{all but finitely many } a_{ij} - \delta_{ij} \text{ are } 0 \end{array} \right\}.$$

(4.1) Show that for all $M \in \mathfrak{gl}_\infty$, $\exp(M) = 1 + M + M^2/2 + \dots$ is well-defined and

$$\exp(M) \in \mathrm{GL}_\infty.$$

(4.2) Show that all $A \in \mathrm{GL}_\infty$ can be written as $\exp(M)$ for some $M \in \mathfrak{gl}_\infty$.

Hint: think about the finite-dimensional case.

(5) Recall that a matrix $M \in \mathfrak{gl}_\infty$ acts on a semi-infinite monomial $v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \ldots$ as follows,

$$\begin{aligned} M \cdot (v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \ldots) := & (M \cdot v_{s_0}) \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \ldots \\ & + v_{s_0} \wedge (M \cdot v_{s_{-1}}) \wedge v_{s_{-2}} \wedge \ldots \\ & + v_{s_0} \wedge v_{s_{-1}} \wedge (M \cdot v_{s_{-2}}) \wedge \ldots \\ & \ldots \end{aligned}$$

Show that the exponent of this action takes the following from,

$$\begin{aligned} & \exp(M) \cdot (v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \ldots) \\ & \qquad = \exp(M) \cdot v_{s_0} \wedge \exp(M) \cdot v_{s_{-1}} \wedge \exp(M) \cdot v_{s_{-2}} \wedge \ldots, \end{aligned}$$

such that the action of M^k is defined via $M \cdot (M \cdot (\ldots))$, and M acts as above. *Hint: think about the finite-dimensional case.*