## EXERCISE SHEET 8

(1) Let $V = \bigoplus_{i=1}^n \mathbb{C}v_i$ be a finite-dimensional vector space. Consider its $k$-wedge power $\wedge^k V$. Recall the Plücker coordinates $a_I$ of a vector $w$ in $\wedge^k V$,

$$w = \sum_{\substack{I=(i_1,\ldots,i_k)\\ i_1<\ldots<i_k}} a_I e_{i_1} \wedge \ldots \wedge e_{i_k},$$

i.e., these are simply the coordinates provided by the basis $\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}$.

(1.1) Consider a $k$-wedge $w = w_1 \wedge \ldots \wedge w_k$ for some $w_i = \sum_i a_{ij} v_j \in V$. To such $w$, we associate a $k \times m$ given by $k$ vectors $w_i$,

$$A = (a_{ij}).$$

Show that the Plücker coordinates $a_I$ of such $w$ are given by the determinants of the $k \times k$ minors of the matrix $A$.

(1.2) Let $1 \leq j_1 < \ldots < j_{k-1} \leq n$ and $1 \leq j'_1 < \ldots < j'_{k+1} \leq n$ be two collections of indices. Show that

$$\sum_{s=1}^{k} (-1)^s \begin{vmatrix} a_{1j_1} & \cdots & a_{1j_{k-1}} & a_{1j'_s} \\ \vdots & \vdots & \vdots & \vdots \\ a_{kj_1} & \cdots & a_{kj_{k-1}} & a_{1j'_s} \end{vmatrix} \cdot \begin{vmatrix} a_{1j'_1} & \cdots & a_{1j'_{s-1}} & a_{1j'_{s+1}} & a_{1j'_{k+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{kj'_1} & \cdots & a_{kj'_{s-1}} & a_{kj'_{s+1}} & a_{1j'_{k+1}} \end{vmatrix} = 0.$$

*Hint: arrange the above sum in the way that the summands are given by determinants of matrices with repeated rows.*

(1.3) Deduce the *Plücker relations* for the vector $w$,

$$\sum_{s=1}^{k+1} (-1)^s a_{j_1,\ldots,j_{k-1},j'_s} \cdot a_{j'_1,\ldots,j'_{s-1},j'_{s+1},\ldots,j'_{k+1}} = 0,$$

such that coefficients $a_{j_1,\ldots,j_{k-1},j'_s}$ are alternating in their indices, e.g., $a_{j_1,\ldots,j_{k-1},j'_s} = -a_{j_1,\ldots,j'_s,j_{k-1}}$.

(2) Let $V$ be a finite dimensional vector space as before. Let $\hat{v}_i$ and $\check{v}_i^*$ be wedging and contraction operators acting on $\bigoplus_{k=0}^n \wedge^k V$, defined in the same way as in Exercise sheet 7. Given a vector $w \in \wedge^k V$, show that the equation

$$\sum_{i=1}^{n} \hat{v}_i(w) \otimes \check{v}_i^*(w) = 0$$

is satisfied, if and only if the corresponding Plücker coordinates $a_I$ satisfy the Plücker relations (for all pairs of collections of indices).

(3) Recall the orbit of the action of $\mathrm{GL}_\infty$ on the vacuum vector $\psi_0$,

$$\Omega \subset F^{(0)}.$$

Let $\mathbb{P}\Omega := \Omega/\mathbb{C}^*$, i.e., the set of vectors in $\Omega$ up to the $\mathbb{C}^*$-scaling, $v \sim \lambda v$. Let $\mathrm{Gr}_\infty$ be the set of all vector subspaces $U$ of $V = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}v_i$, such that $U$ contains $\bigoplus_{i\leq -k} \mathbb{C}v_i$ for $k \gg 0$ as a subspace of codimension $k$. Construct a natural bijection between $\mathbb{P}\Omega$ and $\mathrm{Gr}_\infty$,

$$f\colon \mathbb{P}\Omega \xrightarrow{\sim} \mathrm{Gr}_\infty.$$

*Hint: think about a finite-dimensional Grassmannian from the lecture notes.*

(4) Recall the Hirota bilinear notation from the lecture notes,

$$Pf \cdot g,$$

for a polynomial $P$ in variables $x_i$.

(4.1) Show that for $P = x^n$,

$$Pf \cdot g = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\partial^k f}{\partial x^k} \frac{\partial^{n-k} g}{\partial x^{n-k}}.$$

(4.2) Show that

$$Pf \cdot f \equiv 0,$$

if and only if $P(x) = -P(-x)$.

(5) Show by a direct computation that

$$h(x, y, t) = \frac{(u-v)^2}{2[\cosh(\frac{1}{2}((u-v)x + (u^2-v^2)y + (u^3-v^3)t + c))]^2}$$

satisfies the KP equation,

$$\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial t} - \frac{3}{2}h\frac{\partial h}{\partial x} - \frac{1}{4}\frac{\partial^3 h}{\partial x^3}\right) - \frac{3}{4}\frac{\partial^2 h}{\partial y^2} = 0,$$

where $(u, v, c)$ are scalar parameters.

(6) Determine the term at $y_r y_1$ of the KP hierarchy from Theorem 10.5 of the lecture notes.