INFINITE-DIMENSIONAL LIE ALGEBRAS AND INTEGRABLE SYSTEMS

Contents

1. Introduction

1.1. **What are they?** Let us start by giving a very rough explanation of two terms that appear in the title of this course. Lie algebras capture a very natural notion - the one of a symmetry. More precisely, an infinitesimal symmetry. The idea of Integrable systems, on the other hand, is more intricate to explain. At a first approximation, it means a differential equation that can be solved exactly.

$$\text{Lie algebras} = \text{Infinitesimal Symmetries}$$

$$\text{Integrable systems} \approx \text{Exactly solvable differential equations}$$

However, such characterisation is neither precise nor does it capture the essence of Integrable systems. In fact, there is no mathematical definition of Integrable systems that would encapsulate the full richness of their world. Nevertheless, there are several symptoms for a differential equation to be an integrable system:

- existence of many symmetries (conserved quantities),
- ability to give explicit solutions,
- presence of algebraic geometric (polynomials).

The properties above are still quite vague and subject to interpretation depending on the example. So, perhaps, the best way to understand what an Integrable system is to show one - this is the main objective of the course. In fact, we will be mainly concerned with one particular Integrable system (and maybe a few others which are very closely related to it). Namely, the Kadomtsev–Petviashvili (KP) equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right) - \frac{3}{4}\frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x,y,t) \in C^\infty(\mathbb{R}^3).$$

This is a non-linear[1] partial differential equation with two space variables and one time variable. In physics, it arises in various ways, but most prominently as a model for shallow-water waves (i.e., when the wavelength of waves is much greater than the

[1]Due to the term $\frac{3}{2}u\frac{\partial u}{\partial x}$.

depth of waters). Its attractiveness from a mathematical point of view stems from the fact that it perfectly satisfies all expectations one would have for an integrable system. It is therefore considered the role model for all Integrable systems with infinitely many degrees of freedom (i.e., its solutions are functions, not points of some affine space $\mathbb{R}^n$).

1.2. **What this course is about?** It is not obvious at all from the first glance, but the KP equation possesses a lot of symmetries. In fact, so many that it is completely characterised by them. In the same way that a circle is completely and uniquely characterized (up to the radius) as the shape that is invariant under rotations.

In particular, by uncovering these symmetries, we will be able to provide various explicit solutions to it, among which are n-solitons. This is where the theory of Lie algebras and Integrable systems intersect, as the former is the language for the symmetry transformations of the latter.

Aim: Understand the KP equation through representation theory.

Our approach will be almost completely algebraic. As a consequence, this course is about:

- Representation theory of Lie algebras,
- Combinatorics,
- A little bit of algebraic geometry,
- A tiny bit of PDE.

And it is not about:

- Functional analysis,
- PDE,
- Mathematical physics.

The approach presented in this course comes late in the history of the KP equation. And arguably it would not be possible without the immense body of work that had existed prior to that and did not involve any representation theory. Unfortunately, we will not cover many other aspects of the KP equation and Integrable systems in general. In particular, tools like Lax pairs, Inverse-scattering method, Spectral curves, etc., will be either completely ignored or mentioned very briefly.

The representation-theoretic approach to the KP equation was discovered by Sato [Sat81] and developed by Date–Jimbo–Kashiwara–Miwa [DJKM81, DJKM82]. It will take us most of the course to set up the right language. We believe that the best source for this are the lecture notes by Kac [KR87]. However, we will also compliment them by the book written by Date, Jimbo and Miwa themselves [MJD00].

1.3. **Overview of the course.** The course will roughly consist of four parts.

1.3.1. *Lie algebras and representation theory.* The first part will be about Lie algebras and their representation theory. After recalling the basic notions of Lie algebra theory, we will study in more detail three infinite-dimensional Lie algebras:

(1) Heisenberg algebra $\mathfrak{Heis}$,
(2) Virasoro algebra $\mathfrak{Vir}$,
(3) Algebra of infinite matrices $\mathfrak{a}_\infty$.

The last[2] is the Lie algebra of symmetry transformations of the KP equation,

$$\mathfrak{a}_\infty = \{(a_{ij})_{i,j\in\mathbb{Z}} \mid \; a_{ij} = 0, \text{ if } |i-j| \gg 0\}.$$

All three algebras are intimately related, and will need to know a few things about all of them (but less for the Virasoro). The main result of this part will be so-called *Boson-Fermion correspondence*, which is a natural isomorphism of two representations of $\mathfrak{a}_\infty$,

$$\mathbb{C}[z^\pm, x_1, x_2, \dots] \cong \wedge^\infty V.$$

The left-hand side are Bosons - symmetric tensors of a vector space with a countable basis, also known as polynomials with infinitely many variables. The right-hand side are Fermions - antisymmetric tensors, also known as the infinite wedge. This correspondence will allow us to translate something complicated on the Boson side to something much simpler on the the Fermion side. This "something complicated" on the Boson side will include the KP equation. To do it, we will need a few combinatorial tools, which is the subject of the second part.

1.3.2. *Combinatorics.* In the second part, we will cover some basic combinatorial tools, like:

- Partitions,
- Schur polynomials,
- Young tableaux,
- Maya diagrams.

They will provide the necessary language to pass between Bosons and Fermions, as well as the building blocks for understanding the KP equation.

1.3.3. *Integrable systems.* In the third part, we will analyse the KP equation. Firstly, we will show that $\mathfrak{a}_\infty$ is the algebra of symmetry of the KP equation, in the sense that it acts on the space of its solutions. The KP equation lives on the Boson side, while its Fermionic counterpart is something more elementary - a quadratic equation similar to a *Plücker relation* in finite dimensions.

Using this, we will construct two kinds of explicit solutions:

- Rational solutions,
$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log(S_\lambda(x,y,t,c_4,c_5,\dots)),$$
where S_λ is a Schur polynomial associated to a partition λ written in the basis of power symmetric polynomials.

- Solitons, e.g.,
$$u(x,y,t) = \frac{1}{2\cosh(\frac{1}{2}(x+y+t))^2}.$$

Moreover, we will describe the space of its solutions explicitly via a certain Infinite-dimensional Grassmannian (Sato Grassmannian),

$$\mathbb{G}\mathrm{r} = \{\text{solutions of the KP equation}\}.$$

[2]To be more precise, its central extension.

We will thereby show that the KP equation possesses all properties mentioned in Section 1.1.

We will also briefly mention other aspects of the KP equation and how they are related to the approach presented in the course. Namely, the Lax pairs and the Inverse-scattering method.

1.3.4. *Further applications.* Lastly, we will talk about some applications of the theory. A particularly beautiful and quite accessible direction is the Hurwitz theory. Hurwitz theory counts (ramified) covers of a sphere. The resulting numbers have deep representaiton-theoretic and algebro-geometric meanings. This will be based on a very short paper by Okounkov [Oko00].

1.4. **Symmetries of equations.**

1.4.1. *Group symmetries.* The best way to end the introduction is to explain what one means by symmetries of an equation. To this end, one cannot think of anything more symmetric than a circle (in fact, in some sense, it is a baby version of the KP equation),

$$x^2 + y^2 = r^2. \tag{1}$$

Let GL_2 be the group of invertible 2×2 matrices with real coefficients. We say that $A \in \mathrm{GL}_2$ is a symmetry of (1), if

$$\text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \in (1), \text{ if and only if } \begin{pmatrix} x \\ y \end{pmatrix} \in (1).$$

It is easy to see that

$$\{A \mid A \text{ is a symmetry of } (1)\}$$

is a subgroup of GL_2. Moreover, it is exactly the orthogonal group O_2, and its elements are given by rotations and reflections

$$T(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

1.4.2. *Infinitesimal symmetries.* Let us define a vector $\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}$ as follows:

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

By differentiating with respect to θ, we obtain that $\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}$ solves the following differential equation:

$$\frac{d}{d\theta} \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}. \tag{2}$$

By solving the differential equation (2), we also recover the rotation matrix $T(\theta)$. In particular, the matrix $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ can be viewed as the first-order approximation of $T(\theta)$. More precisely, we have the following relation between these matrices,

$$e^{\theta T} = I_2 + \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\theta^2}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 + \cdots = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Moreover, the fact that $T(\theta)$ is a symmetry of the circle (1) can be rephrased in terms of (2): If the initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$ solves (1), then the solution of (2) given by $\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}$ solves (1) for all θ. While this seems just a more complicated way to say the same thing, this viewpoint generalises much better than the one from Section 1.4.1. Overall, we can rightfully call the matrix T as a infinitesimal symmetry of (1).

1.4.3. *Structure of infinitesimal symmetries.* If

$$\{\text{Symmetries of (1)}\} = \text{Group},$$

what structure does

$$\{\text{Infinitesimal symmetries of (1)}\}$$

possess? Unfortunately, at this stage we exhausted the capacity of (1) to illustrate the full depth of these notions, as in this case, infinitesimal symmetries are essentially a vector space spanned by T. Hence let us consider a more general situation by taking exponents of an arbitrary matrix. Let

$$M_1, M_2 \in \mathfrak{gl}_n = \{n \times n \text{ matrices}\}$$

be two matrices. Consider their exponents e^{M_1} and e^{M_2}, which are invertible matrices. The Baker–Campbell–Hausdorff formula says that the product of e^{M_1} and e^{M_2} can be expressed as follows,

$$e^{M_1} \cdot e^{M_2} = e^{M_1 + M_2 + [M_1, M_2] + \frac{1}{12}[M_1 - M_2, [M_1, M_2]] + \cdots},$$

where

$$[M_1, M_2] = M_1 \cdot M_2 - M_2 \cdot M_1.$$

In particular, if we treat M_1 and M_2 as the first-order terms of e^{M_1} and e^{M_2}, then $[M_1, M_2]$ can be interpreted as the first-order version of the matrix multiplication. The vector space $\mathfrak{gl}_n$ together with the bracket $[\ ,\ ]$ is an example of Lie algebra. Hence the answer to the question above is

$$\{\text{Infinitesimal symmetries}\} = \text{Lie algebra}.$$

1.5. **Infinitesimal symmetries of equations with infinite degrees of freedom.** In the previous example, the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ lived in a finite-dimensional space $\mathbb{R}^2$. We now want to replace a two-vector by a two-variable function $u(x, y)$. The equation (1) has a direct generalisation, which is the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + r^2 u = 0, \quad u = u(x, y) \in C^\infty(\mathbb{R}^2).$$

Rotation matrices also act on functions $u(x, y)$ by precomposition,

$$(T(\theta) \cdot u)(x, y) := u\left(T(-\theta)\begin{pmatrix} x \\ y \end{pmatrix}\right),$$

one can readily see that this is also a symmetry of the Laplace equation in the same sense as before. We define an analogue of the vector $\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}$, namely, the function of three variables

$$u(x, y, \theta) := (T(\theta) \cdot u)(x, y).$$

By differentiating and using the composition rule of differentiation, we obtain an analogue of (2),

$$\frac{\partial}{\partial \theta} u(x, y, \theta) = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) u(x, y, \theta) := x\frac{\partial}{\partial y} u(x, y, \theta) - y\frac{\partial}{\partial x} u(x, y, \theta).$$

We see that the linear operator $\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ plays the same role as the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Moreover, by solving the differential equation above we recover the operator $T(\theta)$ acting on functions in the same way as before,

$$e^{\theta\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)} = T(\theta).$$

An important lesson to take away is that differential operators can be viewed as certain infinitesimal symmetries of functions. It is precisely these kinds of symmetries that we will explore and that the KP equation possesses.

2. Lie algebras

2.1. **Definition.** We start with recalling what Lie algebras are. In this lecture, we will be concerned with finite-dimensional Lie algebras. In fact, we will not need anything from the theory of finite-dimensional Lie algebras apart from a few basic notions. Throughout the course, we will be working with Lie algebras over the field of complex numbers $\mathbb{C}$.

Definition 2.1. Let $(\mathfrak{g}, [\ ,\ ])$ be $\mathbb{C}$-linear space with $\mathbb{C}$-bilinear map

$$[\ ,\ ] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

The pair $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra if

(1) $[\ ,\ ]$ is skew-symmetric,
(2) $[\ ,\ ]$ satisfies the Jacobi identity,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \text{for all } x, y, z \in \mathfrak{g}.$$

We call $[\ ,\ ]$ a commutator or Lie bracket. Very often we will drop it from the notation.

2.2. **Examples.**

2.2.1. *Abelian Lie algebras.* The simplest example is an abelian Lie algebra,

$$(V, [\ ,\ ])\quad [x,y]=0 \text{ for all } x,y\in V,$$

where V is some vector space. Despite its apparent simplicity, it is one of the most important kinds of Lie algebras.

2.2.2. *Lie algebras and associative algebras.* An associative $\mathbb{C}$-algebra is a $\mathbb{C}$-vector space with a bilinear product, $(A, -\cdot-)$,

$$-\cdot-\colon A\times A\to A,$$

which satisfies the associativity axiom,

$$(x\cdot(y\cdot z))=((x\cdot y)\cdot z).$$

The vector space A carries a Lie-algebra structure defined as follows,

$$[x,y]=x\cdot y-y\cdot x.$$

The fact that this bracket satisfies the Jacobi identity follows from the associativity axiom. A more explicit example of this construction is given by the algebra of $n\times n$ matrices,

$$\mathfrak{gl}_n=\{n\times n \text{ complex matrices}\}.$$

which is the mother of all Lie algebras. More basis-independently, for a vector space V, the vector space of linear endomorphisms,

$$\mathfrak{gl}(V)=\{f\colon V\to V\},$$

is an associative algebra, such that the product is given by composition of maps. Hence it also carries a Lie bracket by the construction above.

2.2.3. *Classical Lie algebras.* The following classical Lie algebras are the core of the theory of finite-dimensional Lie algebras.

(1) $\mathfrak{sl}_n=\{x\in\mathfrak{gl}_n \mid \operatorname{tr}(x)=0\}$,
(2) $\mathfrak{o}_n=\{x\in\mathfrak{gl}_n \mid x+x^T=0\}$, where x^T is the transpose of a matrix,
(3) $\mathfrak{sp}_{2n}=\{x\in\mathfrak{gl}_n \mid J_n\cdot x+x^T\cdot J_n=0\}$, where $J_n=\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix}$.

By exponentiating matrices above, we land into

(1) Special linear group Sl_n,
(2) Orthogonal group O_n,
(3) Symplectic group Sp_n.

More specifically, $\mathfrak{sl}_2$ admits the following explicit presentation:

$$\mathfrak{sl}_2=\{e,h,f\},\quad h=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, e=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, f=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$
$$[h,e]=2e,\quad [h,f]=-2f,\quad [e,f]=h.$$

2.3. Basic notions.

Definition 2.2. Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra if

$$[x,y]_{\mathfrak{g}} \in \mathfrak{h}, \qquad \text{for all } x, y \in \mathfrak{h}.$$

It is an ideal if

$$[x,y]_{\mathfrak{g}} \in \mathfrak{h}, \qquad \text{for all } x \in \mathfrak{g} \text{ and } y \in \mathfrak{h}.$$

Example 2.3. All classical Lie algebras are Lie subalgebras of $\mathfrak{gl}_n$, so is the abelian Lie algbera of diagonal matrices.

Definition 2.4. A $\mathbb{C}$-linear map $f\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if

$$f([x,y]_{\mathfrak{g}}) = [f(x), f(y)]_{\mathfrak{h}}.$$

Example 2.5. For any Lie algebra $\mathfrak{g}$, there exist a Lie algebra homomorphism,

$$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad x \mapsto [x, -],$$

where $\mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of linear endomorphisms of the vector space $\mathfrak{g}$. The fact that this map is indeed a Lie algebra homomorphism is equivalent to the Jacobi identity.

Lemma 2.6. *If $f\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then*

(1) $\mathrm{kef}(f)$ *is an ideal,*
(2) $\mathrm{Im}(f)$ *is a Lie subalgebra.*

Proof. A simple verification of definitions. □

Definition 2.7. If $\mathfrak{g}$, $\mathfrak{h}$ are Lie algebras, then the direct sum of vector spaces $\mathfrak{g} \oplus \mathfrak{h}$ carries a natural Lie bracket given by

$$[(x,y),(x',y')]_{\mathfrak{g}\oplus\mathfrak{h}} = ([x,x']_{\mathfrak{g}}, [y,y']_{\mathfrak{h}}).$$

We call the resulting Lie algebra a direct sum of Lie algebras.

The notion of direct sums of Lie algebras admits the following generalisation, if one of the Lie algebras is abelian.

Definition 2.8. Given Lie algebras $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{a}$, such that $\mathfrak{a}$ is abelian. We say that $\mathfrak{h}$ is a central extension of $\mathfrak{g}$ by $\mathfrak{a}$ if

(1) there exists a sequence of Lie algebra homomorphisms

$$\mathfrak{a} \xhookrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g},$$

such that $\mathrm{Ker}(\pi) = \mathrm{Im}(\iota)$,
(2) elements of $\mathfrak{a}$ commute with elements of $\mathfrak{h}$, i.e., $[\mathfrak{a}, \mathfrak{h}]_{\mathfrak{h}} = 0$ (identifying $\mathfrak{a}$ with $\mathrm{Im}(\iota)$).

Example 2.9. Consider two abelian Lie algebras,

$$\mathfrak{g} = \{x, y\}, \quad \mathfrak{a} = \{z\}.$$

We define a central extension of $\mathfrak{g}$ by $\mathfrak{a}$ as follows,

$$\mathfrak{h} = \{x,y,z\}, \quad [x,y] = z, [z,x] = [z,y] = 0.$$

The Lie algebra $\mathfrak{h}$ is called (small) Heisenberg algebra. It also admits the following presentation in terms of 3×3 matrices,

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Later, we will define its infinite-dimensional generalisation $\mathfrak{Heis}$ by adding infinitely many pairs of elements x and y with the commutation relation as above.

In the exercise sheet, we show that central extensions are classified by bilinear skew-symmetric maps

$$\Theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a},$$

which satisfy the following property:

$$\Theta(x, [y, z]) + \Theta(y, [z, x]) + \Theta(z, [x, y]) = 0.$$

2.4. **Representations of Lie algebras.**

Definition 2.10. A representation of a Lie algebra is a pair (V, ρ), a vector space V and a Lie algebra homomorphism

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V).$$

Very often we will drop ρ from the notation. We will also write

$$x \cdot v := \rho(x)(v).$$

Lie algebras are understood via their representations. Practically, this means that the study of Lie algebras is very often (if not always) reduced to expressing (or, in other words, representing) them in terms of some matrices. Moreover, in many situations, representations rather than Lie algebras themselves are the objects that we care about. If we view Lie algebras as infinitesimal symmetries, then their representations are realisations of these symmetries acting on some particular space.

Example 2.11. We have already seen an important representation associated to any Lie algebra, namely, the adjoint representation

$$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad x \mapsto [x, -].$$

Example 2.12. Since all classical Lie algebras from Section 2.2.3 are subalgebras of a Lie algebra of matrices, they come with a natural representation given by acting on $\mathbb{C}^n$.

Example 2.13. Consider the Lie algebra $\mathfrak{sl}_2$ from the end of Section 2.2.3. Let

$$V_n = \{x^k y^{n-k} \mid 0 \le k \le n\}$$

be the vector space of homogenous polynomials of total degree n in two variables x and y. There exists an action of $\mathfrak{sl}_2$ on V_n by associating

$$e \mapsto x\frac{\partial}{\partial y}, \quad f \mapsto y\frac{\partial}{\partial x}, \quad h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Definition 2.14. Let V_1 and V_2 be representations of a Lie algebra $\mathfrak{g}$. A homomorphism of representations is a linear map $f\colon V_1 \to V_2$, such that

$$f(x \cdot v) = x \cdot f(v), \qquad \text{for all } x \in \mathfrak{g} \text{ and } v \in V.$$

We say that two representations are isomorphic (or equivalent) if f is bijective.

Definition 2.15. Let V be a representation of a Lie algebra $\mathfrak{g}$. A subpresentation of V is a vector subspace $V' \subseteq V$, such that

$$x \cdot v \in V', \qquad \text{for all } x \in \mathfrak{g} \text{ and } v \in V',$$

a subrepresentation of V is also a representation of $\mathfrak{g}$.

We say that a representation V is irreducible, if it does not have any subrepresentations other than V itself or $\{0\}$.

Lemma 2.16. *If $f\colon V_1 \to V_2$ is a homomorphism of representations, then* $\mathrm{Ker}(f)$ *and* $\mathrm{Im}(f)$ *are subrepresentations.*

Proof. A simple verification of definitions. □

Let us consider the representations V_n of $\mathfrak{sl}_2$ introduced in Example 2.13. We will show that they are irreducible. This will involve some ideas that we will extensively use later.

Lemma 2.17. *Representations V_n of $\mathfrak{sl}_2$ are irreducible.*

Proof. Firstly, by using the definition of these representations, we see that repeated application of the operators e and f on monomials have the following form:

$$e^k \cdot x^{n-k} y^k = k! x^n. \tag{3}$$

$$f^k \cdot x^n = \frac{n!}{(n-k)!} x^{n-k} y^k \tag{4}$$

Given now a non-zero subrepresentation $0 \neq V \subseteq V_n$. Consider an arbitrary non-zero element v of V,

$$v = \lambda x^{n-k} y^k + \ldots$$

where "..." stands for summands whose powers of y are strictly smaller than k (we always can pick a summand with the leading power of y). Then by (3), we get

$$e^k \cdot v = \lambda k! x^n,$$

while by (4), we obtain that for all $0 \leq h \leq n$,

$$f^h \cdot (e^k \cdot v) = \lambda k! \frac{n!}{(n-k)!}\, x^{n-h} y^h.$$

Since V_n is spanned by elements $x^{n-h} y^h$, we obtain that $V = V_n$. Hence V_n is irreducible. □

The element $x^n \in V_n$ is refereed to as the highest-weight vector. While e and f are called raising and lowering operators, respectively. This terminology aims to capture the fact that by acting by e and f we raise and lower the powers of x, and the element with the highest power of x is x^n. Such structure is very often present in the representation theory of Lie algebras.

Note that in the proof above we also slightly abused the notation writing e^k and f^k. Since $\mathfrak{sl}_2$ is not an associative algebra, but a Lie algebra, the only legal operations in $\mathfrak{sl}_2$ are addition and commutation $[\ ,\ ]$. What we really meant is $\rho(e)^k$ and $\rho(f)^k$, which is permitted because $\rho(e)^k$ and $\rho(f)^k$ are endomorphisms of a vector space (or, after choosing a basis, matrices). In fact, one can define an object which formally contains powers of elements of a Lie algbera $\mathfrak{g}$, this will allow us to talk about symbols like e^k and f^k without resorting to specific representations.

Definition 2.18. Given a Lie algebra $\mathfrak{g}$. The universal enveloping algebra associated to $\mathfrak{g}$ is

$$U(\mathfrak{g}) := \bigoplus_{k\geq 0} \mathfrak{g}^{\otimes k}/I, \quad I = \langle x\otimes y - y\otimes x - [x,y]\rangle.$$

More explicitly, this is a tensor algebra quotient by the ideal generated by $x\otimes y - y\otimes x - [x,y]$.[3] It has an associative product given by the tensor multiplication,

$$(x_1\otimes\ldots\otimes x_n)\cdot(y_1\otimes\ldots\otimes y_m) := x_1\otimes\ldots\otimes x_n\otimes y_1\otimes\ldots\otimes y_m.$$

In particular, symbols e^k and f^k from the proof of Lemma 2.17 can be treated as elements of $U(\mathfrak{sl}_2)$. Moreover, given a representation $\rho\colon \mathfrak{g}\to\mathfrak{gl}(V)$, by the virtue of the definition of $U(\mathfrak{g})$, there exists a homomorphism of associative algebras

$$\rho\colon U(\mathfrak{g})\to\mathfrak{gl}(V), \quad x_1\cdot\ldots\cdot x_n\mapsto\rho(x_1)\cdot\ldots\cdot\rho(x_n)$$

where the associative product of $\mathfrak{gl}(V)$ is given by the composition of endomorphisms. Since $\rho\colon\mathfrak{g}\to\mathfrak{gl}(V)$ is a Lie algebra homomorphism, its extension preserves the relation I, hence $\rho\colon U(\mathfrak{g})\to\mathfrak{gl}(V)$ is indeed well-defined. Observe also that $\mathfrak{g}\subset U(\mathfrak{g})$.

3. Heisenberg algebra

3.1. **Definition of Heisenberg algebra.** We are ready to introduce our first infinite-dimensional Lie algebra, Heisenberg algebra,

$$\begin{aligned}\mathfrak{Heis} &:= \{a_n, \mathbb{1} \mid n\in\mathbb{Z}\}\\ [\mathbb{1}, a_n] &= 0\\ [a_n, a_m] &= n\delta_{n,-m}\mathbb{1},\end{aligned}$$

where $\delta_{m,-n}$ is the Kronecker delta function,

$$\delta_{i,j} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise.}\end{cases}$$

Being a central extension of an abelian Lie algebra $\{a_n \mid n\in\mathbb{Z}\}$ by one-dimensional Lie algebra $\{\mathbb{1}\}$, $\mathfrak{Heis}$ is the simplest infinite-dimensional non-abelian Lie algebra. Recall that we already considered its finite-dimensional analogue in Example 2.9.

3.2. **Bosonic representation of Heisenberg algebra.**

[3] If $\mathfrak{g}$ is abelian, then $U(\mathfrak{g}) = \mathrm{Sym}^\bullet(\mathfrak{g})$.

3.2.1. *Definition of B.* The importance of Heisenberg algebra arises through its representation on the space of polynomial with infinitely many variables, referred to as Bosonic Fock space. Let

$$B = \mathbb{C}[x_1, x_2, x_3, \dots] := \{x_1^{k_1} \dots x_n^{k_n} \mid k_j > 0, n \geq 0\},$$

where x_0 is is the constant polynomial 1. Given $\mu \in \mathbb{C}$, then $\mathfrak{Heis}$ acts on B via the following association:

$$\begin{aligned} a_n &\mapsto \frac{\partial}{\partial x_n} \quad \text{if } n > 0 \\ a_n &\mapsto n x_n \cdot \quad \text{if } n < 0 \\ a_0 &\mapsto \mu \\ \mathbb{1} &\mapsto 1. \end{aligned}$$

More explicitly, a_n acts by multiplication with nx_n for negative n, a_0 acts by multiplication with μ, while $\mathbb{1}$ acts by multiplication with 1. The fact that this indeed defines a representation of $\mathfrak{Heis}$ essentially follows from the product rule for differentiation,

$$\frac{\partial}{\partial x_n}(nx_n \cdot p(x_1, \dots, x_k)) - nx_n \cdot \frac{\partial}{\partial x_n} p(x_1, \dots, x_k) = np(x_1, \dots, x_k).$$

The necessity of the central extension can also be appreciated through this construction.

3.2.2. *Irreducibility of B.*

Lemma 3.1. *The representation B of $\mathfrak{Heis}$ is irreducible.*

Proof. The strategy of the proof is exactly the same as in Lemma 2.17. Given any polynomial in $p \in B$, then by acting with a_n for positive n, we can reduce P to the constant polynomial 1. By acting with a_n for negative n on the constant polynomial 1, we can get any polynomial $p' \in B$. This shows that any non-zero subrepresentation of B must B itself. □

The constant polynomial $1 \in B$ plays the same role as x^n in Lemma 2.17. We can also call it a highest-weight vector. However, in the context of the representation B, it is more often referred to as a *vacuum vector*. While a_{-n} and a_n are called creation and annihilation operators, respectively. This terminology originates from physics, where B is the space of states of free bosons. The reason for such terminology is clear: by acting with a_{-n} on 1 we can "create" any polynomial, while by acting with a_n we can "annihilate" polynomials.

Let us list the properties that the constant polynomial $v := 1 \in B$ has,

$$\begin{aligned} a_n \cdot v &= 0 \quad \text{for } n > 0 \\ a_0 \cdot v &= \mu v \\ \mathbb{1} \cdot v &= v. \end{aligned} \tag{5}$$

We now characterise the representation B in terms of the vacuum vector v.

Proposition 3.2. *Let V be a representation of $\mathfrak{Heis}$ with a vector $v \in V$ satisfying (5). Then elements of the form $a_{-1}^{k_1} \dots a_{-n}^{k_n} \cdot v$ for $k \in \mathbb{Z}_{>0}$ are linearly independent. If these monomials span V, then V is isomorphic to B.*

Proof. We have a linear map from B to V, defined as follows:

$$\phi \colon B \to V, \quad \phi(p(x_1, \dots, x_n)) = p(a_{-1}, \dots, \frac{a_{-n}}{n}) \cdot v,$$

where $p(x_1, \dots, x_n)$ is some polynomial in B, and $p(a_{-1}, \dots, \frac{a_{-n}}{n}) \in U(\mathfrak{Heis})$ (see the discussion around Definition 2.18). By construction, ϕ is a homomorphism of representations of B. Moreover, since B is irreducible, and ϕ is not a trivial map, by Lemma 2.16, we must have that $\mathrm{Ker}(\phi) = \{0\}$. This shows that $a_{-1}^{k_1} \dots a_{-n}^{k_n} \cdot v$ are linearly independent.

If these elements span V, then ϕ is surjective. We conclude that in this case ϕ must be an isomorphism. □

3.2.3. *Hermitian form on B.* We will now equip with B with a Hermitian form. To do so, observe that there exists a natural anti-linear involution,

$$\omega \colon \mathfrak{Heis} \xrightarrow{\sim} \mathfrak{Heis}, \quad \omega(\lambda a_n) = \overline{\lambda} a_{-n}, \ \omega(\lambda \mathbb{1}) = \overline{\lambda} \mathbb{1},$$

where λ is a complex number and $\overline{\lambda}$ is its conjugate. The involution ω also extends to the universal enveloping algebra $U(\mathfrak{Heis})$.

Definition 3.3. Define the vacuum expectation value $\langle p \rangle$ of a polynomial $p \in B$ as the constant term of p.

For $p, q \in B$, we then define

$$\langle p \mid q \rangle := \langle \omega(p) \cdot q \rangle,$$

where $\omega(p) \in U(\mathfrak{Heis})$ acts on q by annihilation operators. More concretely, if $p = x_1^{k_1} \dots x_n^{k_n} = a_{-1}^{k_1} \dots a_{-n}^{k_n} \cdot v$, then $\omega(p) = a_1^{k_1} \dots a_n^{k_n}$. It is an exercise to verify that $\langle \mid \rangle$ is a (non degenerate) Hermitian form, such that

$$\langle v \mid v \rangle = 1, \quad \langle x_1^{k_1} \dots x_n^{k_n} \mid x_1^{k_1} \dots x_n^{k_n} \rangle = \prod_{j=1}^{n} k_j! j^{k_j},$$

and, moreover, non-equal monomials $x_1^{k_1} \dots x_n^{k_n}$ are orthogonal.

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