A SHORT SUMMARY FOR THE EXAM PREPARATION

(1) Lie algebras: definitions, basic notions, examples. This includes definitions of a representation, irreducibility of a representation, central extensions, the universal enveloping algebra.
(2) The definition of the Heisenberg algebra $\mathfrak{Heis}$.
(3) The representation of $\mathfrak{Heis}$ on $B = \mathbb{C}[x_1, x_2, \dots]$ (also known as Bosonic Fock space); the proof of irreducibility of B.
    (3.1) The representation B is determined by the vacuum vector, Proposition 3.2 in the lecture notes.
    (3.2) The Hermitian pairing on B.
(4) The definition of Witt and Virasoro algebras, $\mathfrak{Wiit}$ and $\mathfrak{Vir}$.
(5) The representation of $\mathfrak{Vir}$ on B via operators L_k.
(6) The problem of irreducibility of B as a representation of $\mathfrak{Vir}$ from Exercise sheet 3.
(7) Grading on B in terms of $k_1 + 2k_2 + \dots nk_n$, page 18 of the lecture notes.
(8) Lie algebras of infinite matrices $\mathfrak{gl}_\infty$ and $\overline{\mathfrak{a}}_\infty$; examples of infinite matrices, e.g., elementary matrices E_{ij}, shift operators Λ_k and their commutation relations.
(9) Infinite wedge $\wedge^\infty V$ (also known as Fermionic Fock space) and its subspaces $F^{(m)}$, where $V = \oplus_{i \in \mathbb{Z}} \mathbb{C} v_i$; the Hermitian pairing on $\wedge^\infty V$.
(10) Semi-infinite monomials, and the combinatorial properties of the associated index sets $S = \{s_0, s_{-1}, s_{-2}, \dots\}$, e.g., Lemma 5.2 in the lecture notes.
(11) Passing from partitions to sets S and vice versa, Lemma 5.3 in the lecture notes.
(12) The degree grading on $F^{(m)}$.
(13) Maya diagrams associated to semi-infinite monomials (and Young tableaux).
(14) The action of $\mathfrak{gl}_\infty$ on $\wedge^\infty V$; the action is well-defined. The explicit action of elementary matrices E_{ij}.
(15) The action of $\overline{\mathfrak{a}}_\infty$ on $\wedge^\infty V$ is not well-defined due to infinities.
(16) The central extension of $\overline{\mathfrak{a}}_\infty$, denoted by $\mathfrak{a}_\infty$. The new action of matrices on $\wedge^\infty V$ is well-defined (i.e., the infinities are removed), respects the Lie bracket of the central extension $\mathfrak{a}_\infty$.
(17) Heisenberg algebra $\mathfrak{Heis}$ sits inside $\mathfrak{a}_\infty$ via Shift operators Λ_k, Lemma 6.3, and therefore acts on $\wedge^\infty V$.
(18) Boson-Fermion correspondence, Part 1. Vector spaces B and $F^{(0)}$ are naturally isomorphic are representations of $\mathfrak{Heis}$; this follows from Proposition 3.2 and the dimension counts of the graded pieces B_k and $F^{(0)}_k$ (both are equal to the number of partitions of k). The vacuum vector $1 \in B$ is sent to the vacuum vector $\psi_0 \in F^{(0)}$.

(19) Boson-Fermion correspondence, Part 2. Using the Boson-Fermion isomorphism, induce the action of $\mathfrak{gl}_\infty$ on B. Determine which polynomials in B correspond to semi-infinite monomials in $F^{(0)}$, and how elementary matrices E_{ij} act on B. This is done in several steps.

- (19.1) Wedging and contraction operators $\hat{v}_i$ and $\check{v}_i^*$ acting on $\wedge^\infty V$; their interpretation in terms of Maya diagrams: they create or remove black dots.
- (19.2) Expressions of elementary matrices E_{ij} in terms of wedging and contraction operators; derivation of the generating series of wedging and contraction operators through the commutation relations, Theorem 8.1.
- (19.3) Derivation of generating series of elementary matrices E_{ij} through Theorem 8.1, Corollary 8.2.
- (19.4) Definition of Schur polynomials; semi-infinite monomials are sent to Schur polynomials with respect to the Boson-Fermion isomorphism. The end of the Boson-Fermion correspondence.

(20) Integrable systems: the Kadomtsev–Petviashvili equation as the equation of waves in shallow waters. Integrable systems = solvable systems of differential equations with many symmetries.

(21) The Kadomtsev–Petviashvili hierarchy as the Bosonic equation of the orbit. This is done in several steps.

- (21.1) The group GL_∞ as exponentials of matrices in $\mathfrak{gl}_\infty$. The action of the group of GL_∞ on $\wedge^\infty V$.
- (21.2) The equation of the orbit $\Omega = \mathrm{GL}_\infty \cdot \psi_0 \subset F^{(0)}$ in terms of wedging and contraction operators, Theorem 9.2.
- (21.3) The infinite-dimensional analogue of the Plücker relations, which are equations of the orbit of $\mathrm{GL}_n \cdot \psi_0 \subset \wedge^k V$ for a finite-dimensional vector space V. The orbit Ω is an infinite-dimensional Grassmannian.
- (21.4) Translating the Fermionic equation of the orbit Ω using the Boson-Fermion correspondence.
- (21.5) Organizing the Bosonic equation of the orbit Ω using the Hirota bilinear notation, Theorem 10.5; this results in the system of differential equations, called Kadomtsev–Petviashvili hierarchy. A polynomial solves this system, if and only if it is in the orbit $\Omega \subset B$.

(22) The first non-trivial equation of the Kadomtsev–Petviashvili hierarchy is the Kadomtsev–Petviashvili equation, Lemma 11.1.

(23) The difference between τ functions of the Kadomtsev–Petviashvili equation and the associated solutions.

(24) Conclude that, by construction, GL_∞ is the group of symmetries of the Kadomtsev–Petviashvili hierarchy: if τ is a solution, then $A \cdot \tau$ for $A \in \mathrm{GL}_\infty$ is also a solution.

(25) The Kadomtsev–Petviashvili hierarchy is an Integrable system: it possesses an infinite-dimensional group of symmetries; its solutions are given by acting with GL_∞ on $1 \in B$.

(26) This leads to the construction of two families of explicit solutions: Schur solutions and n-solitons. Schur solutions are given by Schur polynomials (after taking log and differentiating twice), they are contained in the orbit by the Boson-Fermion correspondence.

(27) On the other hand, n-solitons are given by acting on 1 with the (u, v)-dependent generating series of elementary matrices, Corollary 12.2, which were derived in the proof of the Boson-Fermion correspondence.

(28) Interpretation of n-solitons are solitary waves packets.