PROBLEMS FOR THE EXAM

(1) Recall the vector space $F^{(0)} \subset \wedge^\infty V$ from the lecture notes. Let $F_5^{(0)} \subset F^{(0)}$ be the subspace of degree 5 elements,

$$F_5^{(0)} = \{\psi \in F^{(0)} |\, \deg(\psi) = 5\}.$$

Calculate the dimension of $F_5^{(0)}$.

(2) Let $\sigma\colon \wedge^\infty \xrightarrow{\sim} \mathbb{B} = [z^\pm, x_1, x_2, \ldots]$ be the Boson-Fermion isomorphism. Determine

$$\sigma(v_2 \wedge v_0 \wedge v_{-2} \wedge v_{-3} \wedge v_{-4} \wedge \ldots).$$

(3) Let $\langle\ |\ \rangle$ be the Hermitian pairing on $B = \mathbb{C}[x_1, x_2, \ldots]$ from the lecture notes. Calculate

$$\langle x^3/6 + x_1 x_2 + x_3 \mid x_1^3/3 - x_3\rangle.$$

Let $S_k(x)$ be defined by

$$\exp\left(\sum_{k\geq 1} x_k z^k\right) = \sum_{k\geq 0} S_k(x) z^k.$$

For $k \geq 4$, calculate

$$\langle S_{k-2}S_2 - S_{k-1}S_1 \mid S_{k-1}S_1 - S_k\rangle.$$

(4) Recall the Lie algebra of infinite matrices with finitely many non-zero entries $\mathfrak{gl}_\infty$. Let $\mathfrak{gl}_\infty \curvearrowright B$ be the action induced by the Boson-Fermion isomorphism. Let E_{ij} be the elementary matrices,

$$(E_{ij})_{km} = \begin{cases} 1, & \text{if } (k,m) = (i,j) \\ 0, & \text{otherwise.} \end{cases}$$

Determine the result of the action of E_{ij} on $1 \in B$,

$$E_{ij} \cdot 1.$$

Determine also the result of the action of E_{ij} on $x_1 \in B$,

$$E_{ij} \cdot x_1.$$

(5) Recall the orbit $\Omega = \mathrm{GL}_\infty \cdot \psi_0 \subset F^{(0)}$, where $\psi_0 = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \ldots$. Determine whether the vector

$$v_7 \wedge v_2 \wedge v_{-2} \wedge v_{-3} \wedge v_{-4} \ldots + v_6 \wedge v_3 \wedge v_{-2} \wedge v_{-3} \wedge v_{-4} \ldots$$

is contained in the orbit.

(6) Consider the following operator acting on $\mathbb{B} = \mathbb{C}[z^{\pm}, x_1, x_2, \ldots]$,

$$\Gamma_0^* = [u^0]\Gamma^*(u) = [u^0]\left(z^{-1}\exp\Big(-\sum_{i\geq 1} u^i x_i\Big)\exp\Big(\sum_{i\geq 1}\frac{u^{-i}}{i}\frac{\partial}{\partial x_i}\Big)\right),$$

where $[u^0](\ldots)$ means that we take the coefficient at u^0. Compute

$$\Gamma_0^* \cdot \left(\sum_{k_1+\ldots+nk_n=k}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_n^{k_n}}{k_n!}\right).$$

(7) Let W be a representation of the Heisenberg algebra $\mathfrak{Heis}$ with a non-zero vector v, such that

$$a_n \cdot v = 0, \quad n > 0$$
$$a_0 \cdot v = v, \quad \mathbb{1} \cdot v = v.$$

Determine

$$\prod_{i=1}^{n} a_i^{k_i} \cdot \prod_{i=1}^{n} a_{-i}^{k_i} \cdot v.$$

(8) Recall the wedging and contraction operators from the lecture notes, $\hat{v}_i$ and $\check{v}_i^*$, acting on $\wedge^{\infty} V$. Consider the following operator,

$$A = \sum_{i>0}\hat{v}_i\check{v}_i^* - \sum_{i\leq 0}\check{v}_i^*\hat{v}_i.$$

Write down the result of its action on the semi-infinite monomial ψ_S with the following index set $S = \{s_0, s_{-1}, s_{-2}, \ldots\}$,

$$s_{-i} = m - i, \qquad \text{for } m > i \geq 0,$$
$$s_{-i} = m - 1 - i, \qquad \text{for } i \geq m.$$

(9) Let $A \in \mathrm{GL}_\infty$ be the matrix defined as follows,

$$A \cdot v_0 = v_3 \quad A \cdot v_{-1} = v_1,$$
$$A \cdot v_3 = v_0 \quad A \cdot v_1 = v_{-1},$$
$$A \cdot v_i = v_i \quad \text{if } i \notin \{-1, 0, 1, 3\}.$$

Draw the Maya diagram of $A \cdot \psi_0$.

(10) Let

$$P(x_1, x_2, x_3, x_4) = S_3(S_1 + S_2) - S_4(1 + S_1).$$

For some $c \in \mathbb{C}$, define

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log(P(x, y, t, c)).$$

Using the Boson-Fermion correspondence, determine whether $u(x, y, t)$ satisfies the KP equation,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{3}{2}u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right) - \frac{3}{4}\frac{\partial^2 u}{\partial y^2} = 0.$$