(1)

(1.1) $$\|A\|^2 \cdot \|B\|^2 = \Big(\sum_{ij} |a_{ij}|^2\Big) \cdot \Big(\sum_{ij} |b_{ij}|^2\Big)$$

$$\|A \cdot B\|^2 = \sum_{ij} \Big| \sum_k a_{ik} b_{kj} \Big|^2$$

$$\wedge|$$

$$\sum_{ij} \sum_k |a_{ik}|^2 \cdot |b_{kj}|^2$$

Need to show:

$$\Big(\sum_{ij} |a_{ij}|^2\Big) \cdot \Big(\sum_{ij} |b_{ij}|^2\Big) - \sum_{ij} \sum_k |a_{ik}|^2 \cdot |b_{kj}|^2 \geqslant 0$$

Expanding the brackets:

$$\sum_{iji'j'} |a_{ij}|^2 \cdot |b_{i'j'}|^2 - \sum_{ij} \sum_k |a_{ik}|^2 \cdot |b_{kj}|^2$$

$$= \sum_{ij\, k \neq k'} |a_{ik}|^2 \cdot |b_{k'j}|^2 \quad *$$

after relabelling indices & cancelling terms

Clearly, $* \geqslant 0$

$$\Rightarrow \quad \|A\| \cdot \|B\| \geqslant \|A \cdot B\|$$

(1.2) This follows from (1.1):

$\forall \varepsilon > 0, \ \exists N$ s.t. $\forall n, m, \ n \geqslant, m \geqslant N$

$$\left\| \sum_{k=m}^{n} \frac{A^k}{k!} \right\| \leq \sum_{k=m}^{n} \frac{\|A^k\|}{k!} \leq \sum_{k=m}^{n} \frac{\|A\|^k}{k!} < \varepsilon$$

by the convergence of the exp. series.

(2) Observe that

$$P \exp(A) P^{-1} = \exp(PAP^{-1})$$

Moreover,
$$tr(PAP^{-1}) = tr(A)$$
$$\det(PAP^{-1}) = \det(A)$$

By the Jordan normal form,

$\exists\, P$ s.t. $\quad PAP^{-1} = D + N, \quad DN = ND$

diagonal $\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$ nilpotent $\begin{pmatrix} 0 & \ddots \\ 0 & 0 \end{pmatrix}$

see next exercise

$$\exp(D+N) = \exp(D) \cdot \exp(N)$$

$$\exp(D) = \begin{pmatrix} \exp(a_1) & & 0 \\ & \ddots & \\ 0 & & \exp(a_n) \end{pmatrix}$$

$$\exp(N) = \begin{pmatrix} 1 & & \ast \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\Rightarrow \det(\exp(D)\cdot\exp(N)) =$$
$$= \prod \exp(a_i)$$

$$tr(D+N) = \sum a_i$$

$$\Rightarrow \exp(tr(D+N)) =$$
$$= \det(\exp(D)\cdot\exp(N)).$$

Since $\exp(a) \neq 0$, $\det(\exp(A)) \neq 0$

(3) (3.1)

Need to show

$$\left(\sum \frac{A^k}{k!}\right)\cdot\left(\sum \frac{B^k}{k!}\right)$$
$$= \left(\sum \frac{(A+B)^k}{k!}\right)$$

Since $AB = BA$, this follows from $\exp(a+b) = \exp(a)\cdot\exp(b)$ by treating a and b as formal commuting symbols

(due to the absolute convergence we can rearrange summands)

(3.2) Take $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\exp(A) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\exp(B) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\exp(A)\cdot\exp(B) \neq \exp(B)\cdot\exp(A)$$

(4) See: Page 67-69: https://www.mathematik.uni-muenchen.de/~schotten/LNP-cft-pdf/04_978-3-540-68625-5_Ch04_23-08-08.pdf