(1)

(1.1) Consider $\phi : V \to V$

$\ker \phi$ and $\operatorname{Im} \phi$ are subreps of V

$\Rightarrow \ker \phi = 0$, $\operatorname{Im} \phi = V$ if $\phi \neq 0$, because V is irreducible

$\Rightarrow \phi$ is bijective and therefore invertible.

(1.2) Since V is irreducible, for $v \neq 0$

$\{ x \cdot v \mid x \in \mathfrak{g} \}$ spans V

$\Rightarrow$ if $\phi(v) = \phi'(v)$, then
$$\phi(x \cdot v) = x \cdot \phi(v) = x \cdot \phi'(v) = \phi'(x \cdot v)$$

$\Rightarrow \phi$ and ϕ' agree on a spanning set of V

$\Rightarrow \phi(v) = \phi(v')$.

(1.3) Choose $v \neq 0 \in V$, then

$$\operatorname{End}_{\mathfrak{g}}(V) \hookrightarrow V$$
$$\phi \mapsto \phi(v)$$

is a linear map which is injective by (1.2).

$\Rightarrow \operatorname{End}_{\mathfrak{g}}(V)$ is of countable dimension because V is so by assumption

(1.4) Assume $D \neq \mathbb{C}$, then since $\mathbb{C}$ is algebraically closed, $\exists x \in D$

* s.t. $\nexists P(t) \in \mathbb{C}[t]$, $P(x) = 0$.
$\neq 0$

Consider now $\left\{ \frac{1}{\lambda - x} \mid \lambda \in \mathbb{C} \right\} = B$

Claim: elements of B are linearly independent

Assume the contrary, then $\exists \alpha_i \in \mathbb{C}$ s.t. $\sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i - x} = 0$

multipling by $\prod_{i=1}^{n} (\lambda_i - x)$, we obtain that x must satisfy a polynomial eq. which contradicts * $\Rightarrow$ B is lin. indep.

Moreover, B is uncountable because $\mathbb{C}$ is uncountable and $B \subseteq D$, because D is a division alg.

$\Rightarrow$ D contains a vector subspace of uncountable dim

$\Rightarrow$ contradiction to the countability of D.

$\Rightarrow D = \mathbb{C}$.

(2) If $c \in \mathfrak{g}$ is central, then

$\forall$ representations V, the linear map $c \cdot - : V \to V$ is a map of representations. Indeed,

$$c \cdot (x \cdot v) - x \cdot (c \cdot v) = [c, x] \cdot v = 0$$

$$\Rightarrow c \cdot (x \cdot v) = x \cdot (c \cdot v).$$

In particular, if V is irreducible, then by (1), $c \cdot -$ must be a multiplication by a scalar, since $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}\,\mathrm{id}$.

(3)

Need to show $\phi(a_k \cdot p(x_1, \ldots, x_n))$

$$= a_k \cdot p\left(a_{-1}, \ldots, \frac{a_{-n}}{n}\right) \cdot v$$

(for $\mathbb{1} \in \mathrm{Heis}$, this is clear).

$j > 0$ $\qquad a_{-j} = j\, x_j$

$$a_{-j} \cdot p(x) = j x_j \cdot p(x)$$

$$\phi(j x_j\, p(x)) = \frac{j\, a_{-j}}{j} \cdot p(a_{-i}/i) \cdot v$$
$$= a_{-j} \cdot p(a_{-i}/i) \cdot v$$

$j = 0$ $\qquad a_0 = \mu \cdot$

$$a_0 \cdot p(x) = \mu\, p(x)$$

$$\phi(\mu\, p(x)) = \mu\, p(a_{-i}/i) \cdot v$$
$$= a_0 \cdot p(a_{-i}/i) \cdot v$$

(since a_0 commutes with all a_k).

$j > 0$ $\qquad a_j = \frac{\partial}{\partial x_j}$

$$\frac{\partial}{\partial x_j} x_1^{\kappa_1} \cdots x_n^{\kappa_n} = \kappa_j\, x_1^{\kappa_1} \cdots x_j^{\kappa_j - 1} \cdots x_n^{\kappa_n}$$

$$a_j \cdot a_{-1}^{\kappa_1} \cdots \frac{a_{-n}^{\kappa_n}}{n^{\kappa_n}} = a_{-1}^{\kappa_1} \cdots a_j \frac{a_{-j}^{\kappa_j}}{j^{\kappa_j}} \cdots \frac{a_{-n}^{\kappa_n}}{n^{\kappa_n}}$$

$$= a_{-1}^{\kappa_1} \cdots a_{-j}\, a_j \frac{a_{-j}^{\kappa_j - 1}}{j^{\kappa_j}} \cdots \frac{a_{-n}^{\kappa_n}}{n^{\kappa_n}}$$

$$+ a_{-1}^{\kappa_1} \cdots \frac{a_{-j}^{\kappa_j - 1}}{j^{\kappa_j - 1}} \cdots \frac{a_{-n}^{\kappa_n}}{n^{\kappa_n}}$$

commutation relation $[a_j, a_{-j}] = j\mathbb{1}$

commutation relation $[a_j, a_{-i}] = 0$, $i \neq j$

$\Downarrow$ by induction

$$a_{-1}^{\kappa_1} \cdots a_{-j}^{\kappa_j}\, \cancelto{0}{a_j} \cdots a_{-n}^{\kappa_n}$$

$$\cdots \frac{\quad}{n^{k_n}}$$

$$+ \, k_j \, a_{-1}^{k_1} \cdots \frac{a_{-j}^{k_j - 1}}{j^{k_j - 1}} \cdots \frac{a_{-n}^{k_n}}{n^{k_n}}.$$

$$\Rightarrow a_j \cdot a_{-1}^{k_1} \cdots \frac{a_{-n}^{k_n}}{n^{k_n}} = k_j \, a_{-1}^{k_1} \cdots \frac{a_{-j}^{k_j - 1}}{j^{k_j - 1}} \cdots \frac{a_{-n}^{k_n}}{n^{k_n}}$$

$$= p\left(a_{-1}, \ldots \frac{a_{-n}^{k_n}}{n^{k_n}}\right)$$

$$\Downarrow \text{ for } p = k_j \, x_1^{k_1} \cdots x_j^{k_j - 1} \cdots x_n^{k_n}$$

$$\Phi(a_j \cdot p(x)) \qquad \text{for } j > 0.$$

$$= a_j \cdot p(a_{-i}/i) \cdot v$$

(4) By construction it is clear that $\langle \mid \rangle$ is linear in the first entry and antilinear in the second.

Since monomials $x_1^{k_1} \ldots x_n^{k_n}$ span the vector space B, to show that $\langle \mid \rangle$ is hermitian it suffices to establish that monomials are orthogonal.

Consider $\langle x_1^{k_1} \ldots x_n^{k_n} \mid x_1^{k_1'} \ldots x_n^{k_n'} \rangle$

$\overset{\text{defn}}{=}$ Constant term of

$$\underbrace{\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{1}{n^{k_n}} \frac{\partial^{k_n}}{\partial x_1^{k_n}} \cdot x_1^{k'_1} \cdots x_n^{k'_n}}_{\star}$$

The expression $\star$ has a constant term iff it is a constant iff

$$k_i = k'_i \quad \forall i$$

In particular, it is zero otherwise.

In the case $k_i = k'_i \ \forall i$

$$\star = \prod_{i=1}^{n} k_i! \; i^{-k_i}$$

a simple calculation.