(1) $d_0^j v = \sum_{i=1}^{m} d_0^j w_i = \sum_{i=1}^{m} \lambda_i^j w_i$

$$\Rightarrow \begin{pmatrix} v \\ d_0 v \\ \vdots \\ d_0^{m-1} v \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_m \\ \lambda_1^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$$

Vandermonde matrix A

$\det(A) \neq 0$ iff all λ_i are distinct.

In our case, $w_i \in V_{\lambda_i}$ for distinct λ_i by construction.

$\Rightarrow \det(A) \neq 0$, A is invertible

$$\Rightarrow \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} v \\ \vdots \\ d_0^{m-1} v \end{pmatrix}$$

which is what we wanted to prove.

Since $V' \subseteq V$ is a subrep.,

$d_0^j \cdot V' \subseteq V'$

$\Rightarrow d_0^j v \in V' \Rightarrow w_i \in V'$

$$\Rightarrow V' = \bigoplus_{\lambda \in \mathbb{C}} (V' \cap V_\lambda) = \bigoplus_{\lambda \in \mathbb{C}} V'_\lambda$$

$(\Leftarrow)$

(2) First, $[d_0, d_m] = -m d_m$

(Virasoro commutation relation).

$$\Rightarrow d_0 \cdot (d_{-i_k} \cdots d_{-i_1} \cdot v)$$

$$= d_{-i_k} \cdot d_0 \cdot d_{i_{k-1}} \cdot \cdots \cdot d_{-i_1} \cdot v$$

$$+ i_k d_{-i_k} \cdots d_{-i_1} \cdot v$$

By induction, we get

$$d_{-i_k} \cdots d_{-i_1} \cdot d_0 \cdot v$$

$$+ \left(\sum_{j=1}^{k} i_j\right) \cdot d_{-i_k} \cdots d_{-i_1} \cdot v$$

$$= \left(h + \sum_{j=1}^{k} i_j\right) \cdot d_{-i_k} \cdots d_{-i_1} \cdot v$$

i.e., $d_{-i_k} \cdots d_{-i_1} v$ is an eigenvector of d_0 with an eigenvalue of the form $h+k$, $k \in \mathbb{Z}_{\geq 0}$

By definition, a highest weight rep. V is spanned by $d_{-i_k} \cdots d_{-i_1} \cdot v$

$$\Rightarrow V = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} V_{h+j}$$

Moreover, since $d_0 \cdot d_i \cdot v = d_i d_0 \cdot v - i d_i v$

$$= (h-i) d_i v$$

for $i > 0$ $\quad h - i \notin \{ h+j \mid j \in \mathbb{Z}_{\geq 0} \}$

$$\Rightarrow d_i v = 0.$$

(3). Assume V has another vector

$0 \neq w \in V$ s.t. $d_i \cdot w = 0$ for $i > 0$

We may assume $w \in \bigoplus_{j>0} V_{h+j}$ by substracting a multiple of the highest weight vector v.

Consider now $V' = \{ d_{-i_k} \cdots d_{-i_1} \cdot w \mid 0 \leq i_1 \leq \cdots \leq i_k \}$.

Claim: V' is a subrep of V

Proof: Need to show $d_j \cdot d_{-i_k} \cdots d_{-i_1} \cdot w \in V'$ $\ast$
(for $j \leq 0$ it is clear, assume $j > 0$).

By applying commutation relations:

$$\ast = d_{-i_k} d_j \cdots d_{-i_1} \cdot w + (j + i_k) d_{j-i_k} \cdots d_{-i_1} \cdot w + \frac{j^3 - j}{12} \delta_{j,i_k} C .$$

By induction, "moving d_j to the right" and using $d_j \cdot w = 0$ $j > 0$, we obtain $\ast$ is expressible in terms of elements in V'.

$\Rightarrow$ V' is a subrep of V.

Moreover, $v \notin V'$, because by the proof of (2), $d_{-i_k} \cdots d_{-i_1} \cdot w$

$$\in \bigoplus_{j>0} V_{n+j}$$

$\Rightarrow V \neq V' \Rightarrow V'$ is a proper subrep.

$\Rightarrow V$ is not irreducible.

Assume now V is not irreducible.

take a subrep $0 \neq V' \subset V$.

By the exercise (1) $V' = \bigoplus_{j \in \mathbb{Z}} (V' \cap V_{n+j})$

take the smallest j such that

$V'_{n+j} := V' \cap V_{n+j} \neq \emptyset$.

Claim: $w \in V'_{n+j}$, $d_i \cdot w = 0$ $i > 0$.

Indeed. $d_0(d_i \cdot w) = (n+j-i)\, d_i w$

But by the assumption on the minimality of j $d_i w = 0$.

$\Rightarrow w \in V'_{n+j}$ for the minimal j is the singular vector.

(4)

$$(4.1)\quad L'_1 = \sum_{j\geq 0} a'_{-j} a'_{j+1}$$

$$L'_2 = a'^2_1 + \sum_{j\geq 0} a'_{-j} a'_{j+2}$$

$$[L'_1, L'_k] = (1-k) L_{1+k} \qquad k\geq 0$$

$$\Downarrow$$

By induction, if $L_1 \cdot v = 0$, $L_2 \cdot v = 0$
then $L_k \cdot v = 0$ for all $k\geq 1$

$$L'_1 \cdot x_1 = \overset{\mu}{(}\, a'_0 a'_1 + a'_{-1} a'_2 + \dots) \, x_1$$

$$\underset{a'_0 = 0}{=} 0 + x_1 \cdot \frac{\partial}{\partial x_2} x_1 + \dots$$

derivatives w.r.t. x_i, $i \geq 3$.

$$= 0$$

$$L'_2 \cdot x_1 = \frac{\partial^2}{\partial x_1^2} x_1 + \dots$$

derivatives w.r.t. x_i $i\geq 2$.

$$= 0$$

$\Rightarrow$ x_1 is indeed a singular vector.

$$(4.2)\quad L'_1 \cdot (x_1^2/2 + x_2)$$

derivatives w.r.t. x_i $i\geq 3$

$$= \left(-\frac{\partial}{\partial x_1} + x_1 \cdot \frac{\partial}{\partial x_2} + \dots \right) x_1^2/2 + x_2$$

$$= -x_1 + x_1 = 0$$

$$L'_2 \cdot (x_1^2/2 + x_2)$$

$$= \left(\frac{\partial}{\partial x_1^2} - \frac{\partial}{\partial x_2} + \dots \right) x_1^2/2 + x_2$$

$$= 1 - 1 = 0$$

$\Rightarrow (x_1^2/2 + x_2)$ is a singular vector.

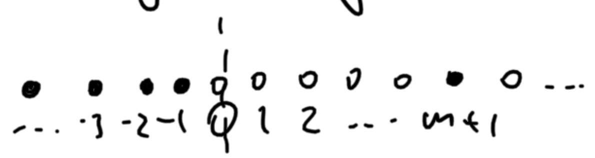
(4.3) Either a direct calculation or use the Boson-Fermion correspondence:

First, identify $B \cong F^{(0)}$, $m \in \mathbb{Z}_{\geq 0}$

With respect to this identification, we have:

1. $a_i \mapsto \lambda_i$ (shift operators)
2. The Schur polynomial S_{m+1} is sent to $w = v_{m+1} \wedge v_{-1} \wedge v_{-2} \wedge \dots$

whose Maya diagram is



$\lambda_i = s_{-i} + i \qquad i \in \mathbb{Z}_{\geq 0}$

$s_0 = m+1 \qquad \lambda_0 = m+1$

$s_{-1} = -1 \qquad \lambda_1 = 0$

$s_{-2} = -2 \qquad \lambda_2 = 0$

$\vdots \qquad\qquad \vdots \qquad\qquad \lambda_k \cdot v_i = v_{i-k}$

$$\lambda_k \cdot (\underbrace{v_{m+1} \wedge v_{-1} \wedge v_{-2}}_{\omega}) \qquad k \neq 0$$

$$= v_{m+1-k} \wedge v_{-1} \wedge \dots$$

$$+ v_{m+1} \wedge v_{-1-k} \wedge \dots$$

$$\dots$$

Virasoro operators become

$$\hat{L}_0 = m^2/4 + \sum_{j \geq 1} \lambda_{-j} \lambda_j$$

$$\hat{L}_1 = -m \lambda_1 + \sum_{j \geq 1} \lambda_{-j} \lambda_{j+1}$$

$$\hat{L}_2 = -m \lambda_2 + \lambda_1^2 + \sum_{j \geq 1} \lambda_{-j} \lambda_{j+2}$$

$\underline{k=1}$

$$-m \lambda_1 \cdot \omega = -m v_m \wedge v_{-1} \wedge v_{-2} \wedge \dots$$

$$\lambda_{-j} \lambda_{j+1} \cdot \omega = v_m \wedge v_{-1} \wedge v_{-2} \wedge \dots \qquad j \leq m$$

$$+ v_{m-j} \wedge v_{-1+j} \wedge v_{-2} \wedge \dots$$

$$\star \qquad + v_{m-j} \wedge v_{-1} \wedge v_{-2+j} \wedge \dots$$

$$\dots$$

- $-m \lambda_1 \cdot \omega$ cancells with m terms

$$v_m \wedge v_{-1} \wedge v_{-2} \wedge \dots$$

- while terms of the form

$$v_{m-j} \wedge v_{-1+j} \wedge v_{-2} \wedge \dots$$

cancel with

$$v_{m-j'} \wedge v_{-1+j'} \wedge v_{-2} \wedge \dots$$

for $j' = m+1-j$

because $v_{m-j} \wedge v_{-1+j} \wedge v_{-2} \wedge \dots$

$$= - v_{m-j'} \wedge v_{-1+j'} \wedge v_{-2} \wedge \dots$$

similarly for other terms in $*$

$\Rightarrow L_1 \cdot w = 0$.

$k=2$. Similar analysis as above.

$\Rightarrow v_{m+1} \wedge v_{-1} \wedge v_{-2} \wedge \dots$ is the singular vector.