(1) It is enough to show this on elementary matrices $E_{ij}$ since they span $\mathfrak{gl}_\infty$.

$$\langle \rho(E_{ij}) \cdot \psi_S | \psi_{S'} \rangle = \begin{cases} \pm 1 & \text{if } S \text{ and } S' \text{ differ by } i \text{ and } j \\ 0 & \text{otherwise.} \end{cases}$$

i.e. $S \setminus \{i\} = S \setminus \{j\}$.

In this case,

same sign as before.

$$\langle \psi_S | \rho(E_{ji}) \psi_{S'} \rangle = \begin{cases} \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \langle \rho(E_{ij}) \cdot \psi_S | \psi_{S'} \rangle$$
$$= \langle \psi_S | \rho(E_{ji}) \cdot \psi_{S'} \rangle.$$

(2) (2.1) Let $V \subseteq F^{(m)}$ be a subrep.

Consider its orthogonal complement $V^\perp = \{ v \in F^{(m)} \mid \langle v | v' \rangle = 0 \ \forall v' \in V \}$.

We want to show that $V^\perp$

is also a subrep of $\mathcal{F}^{(m)}$,

i.e. $M \cdot v \in V^\perp$ for $\forall v \in V^\perp$ & $\forall M \in gl_\infty$.

$$\langle Mv | v' \rangle = \langle v | M^\dagger v' \rangle$$

$v' \in V$, $M^\dagger \cdot v' \in V$, $v \in V^\perp$

$\Rightarrow \langle Mv | v' \rangle = 0$ for $\forall\, v' \in V$

$\Rightarrow Mv \in V^\perp$

$\Rightarrow V^\perp$ is a subrep of $\mathcal{F}^{(m)}$.

Moreover, $\mathcal{F}^{(m)} = V \oplus V^\perp$ as reps of $gl_\infty$.

By induction, we can split $\mathcal{F}^{(m)}$ as sums of irreps.

(2.2) Given a semi-infinite monomial

$$\psi_S = v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \cdots$$

Then

$$\psi_S = \rho(E_{s_{-k}\, m-k}) \cdots \rho(E_{s_0 m}) \cdot \psi_m$$

such that $v_i = i+m \ \forall i < -k$.

(2.3) By (2.1), $F^{(m)} = V \oplus V^{\perp}$ for any subrep $V$.

as reps of $gl_\infty$

Since $\psi_m$ is the unique semi-infinite monomial of degree $\deg(\psi_m)=0$, it must belong to $V$ or $V^{\perp}$.

But by (2.2), all other semi-infinite monomials can be created from it,

$\Rightarrow$ $V = F^{(m)}$ or $V^{\perp} = F^{(m)}$

$V^{\perp} = \{0\}$ $\qquad$ $V = \{0\}$.

$\Rightarrow$ $F^{(m)}$ cannot have any non-trivial subreps.

i.e. it is irreducible.

(3) For $d_n, d_m$ s.t. $n \neq -m$,

we have $[d_n, d_m]_{\bar{a}_\infty} = [d_n, d_m]_{a_\infty}$

in $\bar{a}_\infty$, it can be computed as follows

$$(d_n d_m - d_m d_n)\cdot v_k =$$

$$= (k-m)k\cdot v_{k-m-n} - (k-n)k\cdot v_{k-m-n}$$

$$= (n-m)k\cdot v_{k-m-n}$$

$$= (n-m)\, d_{m+n}\cdot v_k .$$

$$\Rightarrow [d_n, d_m]_{a_\infty} = (n-m)\, d_{m+n}$$

if $m \neq -n$.

Assume now $n = -m$.

$$d_n = \sum_i (i+n) E_{i,i+n}$$

$$[d_n, d_{-n}] = [d_n, d_{-n}]_{\bar{a}_\infty} + \Theta(d_n, d_{-n}) I.$$

$$= 2n d_0 + \Theta(d_n, d_{-n}) I .$$

the central extension cocycle.

$$\Theta\Big(\sum_i (i+n) E_{i,i+n}, \sum_i i\, E_{i+n,i}\Big)$$

$$= \Theta\Big(\sum_{\substack{i\le 0\\ i+n\ge 1}} (i+n) E_{i,i+n}, \sum_{\substack{i\le 0\\ i+n\ge 1}} i\, E_{i+n,i}\Big).$$

on other terms $\Theta$ vanishes

$$= \sum_{0\ge i\ge 1-n} (i+n) i\, \underbrace{\Theta(E_{i,i+n}, E_{i+n,i})}_{=1}$$

$$= \sum_{0 \geq i \geq 1-n} (i+n)i$$

$$= -2 \cdot \frac{n^3-n}{12}$$

Exactly the same computation holds for $d_n^{\alpha,\beta}$.

$\Rightarrow$ operators $d_n$ induce the action of the Virasoro algebra.