(1) $\check{v}_i^*(v_{s_0} \wedge v_{s_{-1}} \wedge \dots) = \check{v}_i^*(v_{s_0})\, v_{s_{-1}} \wedge \dots$

$$- \check{v}_i^*(v_{s_{-1}})\, v_{s_0} \wedge \dots$$

$$+ \dots$$

all summands $= 0$ unless $s_k = i$ for some k

in which case

$$= \pm\, v_{s_0} \wedge v_{s_{-1}} \wedge \dots \wedge v_{s_{k+1}} \wedge v_{s_{k-1}} \wedge \dots$$

where the sign is determined by k

$$= (-1)^k\, v_{s_0} \wedge v_{s_{-1}} \wedge \dots v_{s_{k+1}} \wedge v_{s_{k-1}} \wedge \dots$$

Similarly for $\hat{v}_i(v_{s_0} \wedge v_{s_{-1}} \wedge \dots)$

$$= v_i \wedge v_{s_0} \wedge v_{s_{-1}} \wedge \dots .$$

By moving v_i to its rightfull position we obtain that

$$\hat{v}_i(v_{s_0} \wedge v_{s_{-1}} \wedge \dots) = (-1)^{k-1}\, v_{s_0} \wedge \dots v_{s_k} \wedge v_i \wedge v_{s_{k-1}} \wedge \dots$$

where $s_{k-1} < i < s_k$

and $= 0$ if $i = s_k$.

(2) If $s' \neq s \cup \{i\}$, then

$$\langle \hat{v}_i \cdot \psi_s \mid \psi_{s'} \rangle = 0 = \langle \psi_s \mid \check{v}_i^* \psi_{s'} \rangle$$

where ψ_s and $\psi_{s'}$ are semi-infinite monomials associated to sets S & S'

Assume $S' = S \cup \{i\}$, then

$$\hat{v}_i \cdot \psi_S = (-1)^{k-1} \psi_{S'}$$

such that $s_{k-1} < i < s_k$

while $\check{v}_i^* \psi_{S'} = (-1)^{k-1} \psi_S$

$$\Rightarrow \langle \hat{v}_i \psi_S \mid \psi_{S'} \rangle = \langle \psi_S \mid \check{v}_i^* \psi_{S'} \rangle = (-1)^{k-1}.$$

(3) (3.1) Since $\psi_S = v_{s_0} \wedge v_{s_{-1}} \wedge v_{s_{-2}} \wedge \dots$

$$\check{v}^*_m \cdot \psi_S = 0 \quad \text{for all } m > s_0.$$

$$\Rightarrow \psi^*_{m_i} \cdot \psi_0 = \Big(\sum_{m \geq 0} b_{i,m} \check{v}^*_{m_i+m} \Big) \cdot \psi_0$$

$$= \Big(\sum_{\substack{m \geq 0 \\ m_i + m \leq 0}} b_{i,m} \check{v}^*_{m_i+m} \Big) \cdot \psi_0$$

Note that the sum above involves only finitely many summands

Similarly $\hat{v}_m \cdot \psi_S = 0$ for all $m \ll 0$
because $s_i = i \ \forall i \ll 0$

$$\Rightarrow \psi_{n_i} \cdot \psi_S = \Big(\sum_{n \geq 0} a_{i,n} \hat{v}_{n_i - n} \Big) \cdot \psi_S$$

$$= \Big(\sum_{\substack{n \geq 0 \\ n_i - n \geq K}} a_{i,n} \hat{v}_{n_i - n} \Big) \cdot \psi_S$$

some constant

again finite sum

$\Rightarrow \quad \langle \psi_0 | \Psi_{n_i} \Psi^*_{m_i} \psi_0 \rangle$ is well-defined.

(3.2) $\quad \langle \psi_0 | \Psi_{n_i} \Psi^*_{m_i} \psi_0 \rangle$

$$= \langle \Psi^*_{n_i} \psi_0 | \Psi^*_{m_i} \psi_0 \rangle$$

$\uparrow$ $\hat{v}_i$ and $\check{v}_i^*$ are adjoints.

By the arguments from (3.1),

$$\check{v}^*_{n_i - n} \psi_0 = 0 \quad \text{for } n_i - n > 0$$

$$\& \quad \check{v}^*_{m_i + m} \psi_0 = 0 \quad \text{for } m_i + m > 0$$

$\Rightarrow$ we may assume that

$$\star \quad \begin{aligned} a_{i,n} &= 0 \quad \text{if } n_i - n > 0 \\ b_{i,m} &= 0 \quad \text{if } m_i + m > 0 \end{aligned}$$

Moreover, by the commutation relation from the Exercise sheet 6 we have

$$\star\star \quad \Psi_{n_i} \Psi^*_{m_j} = - \Psi^*_{m_j} \Psi_{n_i} + \underbrace{\sum_{\substack{n,m \\ m_j + m \\ = n_i - n}} a_{i,n} \cdot b_{j,m}}_{\heartsuit} \cdot \mathrm{id}$$

and $\heartsuit = \langle \psi_0 | \Psi_{n_i} \Psi^*_{m_j} \psi_0 \rangle$

because $\quad = \langle \Psi^*_{n_i} \psi_0 | \Psi^*_{m_j} \psi_0 \rangle$

and vectors in $\Psi^*_{n_i} \psi_0$ pair

non-trivially with vectors in $\psi^*_{m_j}\psi_0$
iff $m_j + n = n_j - n$.

Let's now prove Wick's theorem by induction.

The base case $k=1$ is trivially true.
Assume Wick's theorem holds for $k-1$

$$\langle \psi_0 | \psi_{n_1} \cdots \psi_{n_k} \psi^*_{m_k} \cdots \psi^*_{m_1} \psi_0 \rangle =$$

$$\langle \psi_0 | \psi_{n_k} \psi^*_{m_k} \psi_0 \rangle \langle \psi_0 | \psi_{n_1} \cdots \psi_{n_{k-1}} \psi^*_{m_{k-1}} \cdots \psi^*_{m_1} \psi_0 \rangle$$

$$- \langle \psi_0 | \psi_{n_1} \cdots \psi_{n_{k-1}} \psi^*_{m_k} \psi_{n_k} \psi^*_{m_{k-1}} \cdots \psi^* \psi_0 \rangle$$

where we used ☆☆

Continue to slide $\psi^*_{m_k}$ to the left to obtain

☆☆☆ $$\sum_{j=0}^{k-1} (-1)^j \langle \psi_0 | \psi_{n_{k-j}} \psi^*_{m_k} \psi_0 \rangle \cdot$$

$$\cdot \langle \psi_0 | \psi_{n_1} \cdots \widehat{\psi}_{n_j} \cdots \psi_{n_k} \psi^*_{n_{k-1}} \cdots \psi^*_{m_1} \psi_0 \rangle$$

$$+ (-1)^k \langle \psi_0 | \psi^*_{n_k} \psi_{n_1} \cdots \psi_{n_k} \psi^*_{m_{k-1}} \cdots \psi^*_{m_1} \psi_0 \rangle$$

$$= (-1)^k \langle \psi_{n_k} \psi_0 | \cdots \rangle = 0$$

by the adjointness by ☆.

By the induction hypothesis,

$$\langle \psi_0 | \psi_{n_1} \cdots \psi_{n_{k-j}} \cdots \psi_{n_k} \psi^*_{m_{k-1}} \cdots \psi^*_{m_1} \psi_0 \rangle$$

$$= \det_{\substack{i,l=1,\dots,k \\ i\neq k,\ l\neq k-j}} \langle \psi_0 | \psi_{n_l} \psi^*_{m_i} \psi_0 \rangle .$$

Combining this with ☆☆☆ we obtain the claim by noticing that ☆☆☆ is the expansion of the determinant with respect to a line.

(4)

(4.1) Since M has only finitily many non-zero entries $\exp(M)$ is defined as in the finite dimensional case.

(4.2) We may assume we are in the finite-dimensional case, in which case it is true (at least for complex matrices) by taking log of a matrix.

(5) Again, we may assume that we are in the finite-dimensional case, then the claim follows from ↑ (because M .)

expanding (acts non-trivially on finitely many vecs)

$$(1+M+M^2/2!+\dots)v_{s_0} \wedge (1+M+M^2/2+\dots)v_{s-1} \wedge \dots (1+M+M^2/2+\dots)v_{s-n}$$

and collecting terms of the same degree:

$$v_{s_0} \wedge v_{s-1} \wedge \dots$$

$$+ \; Mv_{s_0} \wedge v_{s-1} \wedge \dots + v_{s_0} \wedge Mv_{s-1} \wedge \dots$$

$$+ \; Mv_{s_0} \wedge Mv_{s-1} \wedge v_{s-2} \wedge \dots + v_{s_0} \wedge Mv_{s-1} \wedge Mv_{s-2} \wedge \dots$$

$$+ \; \frac{M^2}{2} v_{s_0} \wedge v_{s-1} \wedge v_{s-2} + v_{s_0} \wedge \frac{M^2}{2} v_{s-1} \wedge v_{s-2}$$

+ terms of order 3

+ .....

$$= (1+M+\frac{M^2}{2}+\dots) \cdot v_{s_0} \wedge v_{s-1} \wedge \dots$$