(1) (1.1) This follows from expanding

$$(\Sigma a_{1j} v_j) \wedge (\Sigma a_{2j} v_j) \wedge \dots \wedge (\Sigma a_{kj} v_j)$$

and collecting coefficients of

$$v_{i_1} \wedge \dots \wedge v_{i_k}.$$

(The proof is induction on k).

(1.1) Expanding the determinant with respect to the final column, we get that

$$\begin{vmatrix} a_{1j_1} & \cdots & a_{1j_{k-1}} & a_{1j'_s} \\ \vdots & \ddots & \vdots & \vdots \\ a_{kj_1} & \cdots & a_{kj_{k-1}} & a_{kj'_s} \end{vmatrix} =$$

$$\sum_{l=1}^{k} (-1)^{k+l} a_{lj'_s} \cdot \underbrace{\begin{vmatrix} a_{1j_1} & \cdots & a_{1j_{k-1}} \\ \vdots & & \vdots \\ a_{l-1j_1} & \cdots & a_{l-1j_{k-1}} \\ a_{l+1j_1} & \cdots & a_{l+1j_{k-1}} \\ \vdots & & \vdots \\ a_{kj_1} & \cdots & a_{kj_{k-1}} \end{vmatrix}}_{= \tilde{a}_l}$$

Then we have to show that

$$\sum_{s=1}^{k+1} (-1)^s \sum_{l=1}^{k} (-1)^{k+l} a_{lj'_s} \cdot \tilde{a}_l .$$

$$\cdot\begin{vmatrix} a_{1j'_1} & \cdots & a_{1j'_{s-1}} & a_{1j'_{s+1}} & \cdots & a_{1j'_{k+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{kj'_1} & \cdots & a_{kj'_{s-1}} & a_{kj'_{s+1}} & \cdots & a_{1j'_{k+1}} \end{vmatrix} = *$$

vanishes. Rearrange the summation

$$= \sum_{l=1}^{k} (-1)^l \tilde{a}_l \cdot \left(\sum_{s=1}^{k+1} (-1)^{k+s} a_{l j'_s} \cdot * \right)$$

but this is an expansion of the determinant in the row of the matrix.

$$\Vert$$

$$\begin{vmatrix} & * & \\ a_{lj'_1} & \cdots \quad \cdots & a_{lj'_{k+1}} \end{vmatrix}$$

this determinant has repeated rows, hence it is zero; the claim then follows.

(1.3) This follows immediately from (1.1) and (1.2).

(2.) Express ω in terms of Plücker coordinates:

$$\omega = \sum_{\underline{I}} a_{\underline{I}} v_{i_1} \wedge \dots \wedge v_{i_k}$$

$$\sum_{i=1}^{n} \hat{v}_i(\omega) \otimes \check{v}_i^*(\omega) =$$

$$= \sum_{\underline{I}, \underline{I}', i} a_{\underline{I}} \cdot a_{\underline{I}'} \; \hat{v}_i(v_{i_1} \wedge \dots \wedge v_{i_k}) \otimes \check{v}_i^*(v_{i'_1} \wedge \dots \wedge v_{i'_k}) \quad \circledast$$

$$\in \Lambda^{k+1} V \otimes \Lambda^{k-1} V$$

Now collect coefficients of the basis elements of $\Lambda^{k+1} V \otimes \Lambda^{k-1} V$

$$(v_{i_1} \wedge \dots \wedge v_{i_{k+1}}) \otimes (v_{i'_1} \wedge \dots \wedge v_{i'_{k-1}}).$$

Note that

$$\hat{v}_i (v_{i_1} \wedge \dots \wedge v_{i_k}) \otimes \check{v}_i^* (v_{i'_1} \wedge \dots \wedge v_{i'_k})$$

$$\shortparallel$$

$$\hat{v}_j (v_{j_1} \wedge \dots \wedge v_{j_k}) \otimes \check{v}_j^* (v_{j'_1} \wedge \dots \wedge v_{j'_k})$$

iff $(i_1, \dots, i_k) \cup \{i\} = (j_1, \dots, j_k) \cup \{j\}$

and $(i'_1, \dots, i'_k) \setminus \{i\} = (j'_1, \dots, j'_k) \setminus \{j\}$.

Hence the coeff in $\circledast$ of

$$(v_{i_1} \wedge \dots \wedge v_{i_{k+1}}) \otimes (v_{i'_1} \wedge \dots \wedge v_{i'_{k-1}}) \qquad \text{is}$$

equal to

$$\sum (-1)^s a_{\underline{I}} \cdot a_{\underline{I}'}$$

where sum over all $\underline{I}$ and $\underline{I}'$

such that

$$\underline{I} \cup \{i_s\} = (i_1, \dots, i_{k+1})$$

and $(i'_1, \dots, i'_{k-1}) = \underline{I}' \setminus \{i_s\}$

But this is exactly equal

to $\sum_{s=1}^{k+1} (-1)^s a_{i_1, \dots, i_{k-1}, i_s} \cdot a_{i'_1, \dots, i'_{s-1}, i_{s+1}, \dots, i_{s+1}}$

Hence we obtain the claim.

(3) By definition,

$$\Omega = \{ \exp(M) \cdot \psi_0 \mid M \in \mathfrak{gl}_\infty \}$$

Recall that $\psi_0 = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$

while $\exp(M) \cdot \psi_0$

$$= \exp(M) \cdot v_0 \wedge \exp(M) v_{-1} \wedge \dots$$

Since M has finitely many non-zero entries, we obtain

☆ that $\exp(M) \cdot v_i = v_i$ for $\forall i \ll 0$.

Let $\quad w_i := \exp(M) \cdot v_i$

Define $U = \langle w_i \rangle$, i.e. the linear span of w_i

By $\star$, U contains $\bigoplus_{i \leq -k} \mathbb{C} v_i$ as a subspace of codimension k

since $\quad U / \bigoplus_{i \leq -k} \mathbb{C} v_i \cong \bigoplus_{i=0}^{-k+1} w_i$

$\Rightarrow$ We associated $U \in Gr_\infty$ to each $\exp(M) \cdot \psi_0$.

Clearly, $\quad \lambda \cdot \exp(M) \cdot \psi_0$ gives the same U for all $\lambda \neq 0$, $\lambda \in \mathbb{C}^*$.

$\Rightarrow$ We obtain a map (of sets)

$$\mathbb{P}(\Omega) \to Gr_\infty$$

The inverse of which is constructed by picking a basis of U, $\{w_0, w_{-1}, \dots\}$ such that $w_{-i} = v_{-i}$ for all $i \geq k$

and associating the

vector $w_0 \wedge w_{-1} \wedge w_{-2} \wedge \ldots$

which is in Ω, because

$\exists\, A \in GL_\infty$ s.t. $A \cdot \psi_0 = w_0 \wedge w_{-1} \wedge \ldots$

Namely A is defined by

$$A \cdot v_i = w_i \; .$$

This establishes the identification

$$\mathbb{P}(\Omega) \xrightarrow{\sim} Gr_\infty \, .$$

(4) (4.1) $P = x^n$

$$P f \cdot g := \frac{\partial^n}{\partial u^n} \left(f(x-u) \cdot g(x+u) \right) \Big|_{u=0}$$

$$\frac{\partial^n}{\partial u^n} \left(f(x-u) \cdot g(x+u) \right)$$

$$= \frac{\partial^{n-1}}{\partial u^{n-1}} \left(-\frac{\partial}{\partial u} f(x-u)\, g(x+u) + f(x-u) \frac{\partial}{\partial u} g(x+u) \right)$$

by induction

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\partial^k}{\partial u^k} f(x-u) \frac{\partial^{n-k}}{\partial u^{n-k}} g(x+u)$$

after setting $u = 0$, we

obtain

$$\sum^{n} (-1)^k \binom{n}{k} \frac{\partial^k}{\partial x^k} f(x) \frac{\partial^{n-k}}{\partial x^{n-k}} g(x) .$$

$k=0 \qquad \binom{n}{k} \partial x \qquad \partial x^{n-k}$

(4.2) Assume $P(x) = -P(-x)$

then

$$P\left(\frac{\partial}{\partial u}\right) f(x-u)\cdot f(x+u)\,|_{u=0}$$

$$= - P\left(-\frac{\partial}{\partial u}\right) f(x-u)\cdot f(x+u)\,|_{u=0}$$

by making the substitution of variables

$u \mapsto -u$, the above

becomes

$$-P\left(\frac{\partial}{\partial u}\right) f(x-u)\cdot f(x+u)\,|_{u=0}$$

$$\Rightarrow \quad P\left(\frac{\partial}{\partial u}\right) f(x-u)\cdot f(x+u)\,|_{u=0} = 0.$$

for all $f(x)$.

$$\Rightarrow \quad Pf\cdot f \equiv 0.$$

Conversely, if $P = x^n$ for n even

then by (4.1), we obtain

that $Pf\cdot f \not\equiv 0$.

Moreover, $Pf\cdot f$ is linear in P,

$$(P+P')f\cdot f = Pf\cdot f + P'f\cdot f,$$

Hence if $-P(-x) \neq P(x)$, then $P(x)$ must have summands of even power.

Let ax^n be the smallest non-zero even term of $P(x)$.

Then $\exists$ polynomial f s.t. $(ax^n) f \cdot f \neq 0$, & since $(P(x) - (ax)^n) f \cdot f$ is of strictly lower degree than $(ax^n) f \cdot f$, we must have that $P f \cdot f \neq 0$.

(5) Direct calculation.

For simplicity, assume $u=1$, $v=0$, $c=0$.

Consider $h(x,y,t) = \dfrac{1}{2\cosh(\frac{1}{2}(x+y+t))^2}$

For brevity, denote

$\cosh := \cosh(\frac{1}{2}(x+y+t))$

$\sinh := \sinh(\frac{1}{2}(x+y+t))$

Then $\dfrac{\partial h}{\partial x} = \dfrac{\partial h}{\partial y} = \dfrac{\partial h}{\partial t} = -\dfrac{1}{2}\dfrac{\sinh}{(\cosh)^3}$

$h\dfrac{\partial h}{\partial} = -\dfrac{1}{} \sinh$

$$\partial x \qquad \overline{4} \; \overline{(\cosh)^5}$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 h}{\partial x^2} = -\frac{1}{4}\cdot\frac{\cosh}{(\cosh)^3} + \frac{3}{4}\frac{(\sinh)^2}{(\cosh)^4}$$

$$\parallel$$

$$-\frac{1}{4}\frac{1}{(\cosh)^2}$$

$$\frac{\partial^3 h}{\partial x^3} = \frac{1}{4}\frac{\sinh}{(\cosh)^3} + \frac{3}{4}\frac{\sinh}{(\cosh)^3} - \frac{3}{2}\frac{(\sinh)^3}{(\cosh)^5}$$

$$= \frac{\sinh}{(\cosh)^3} - \frac{3}{2}\frac{(\sinh)^3}{(\cosh)^5}$$

$$\Rightarrow \quad \frac{\partial h}{\partial t} - \frac{3}{2} h \frac{\partial h}{\partial x} + \frac{1}{4}\frac{\partial^3 h}{\partial x^3}$$

$$= -\frac{3}{8}\frac{\sinh}{(\cosh)^3} = \frac{3}{4}\cdot\frac{\partial h}{\partial x}$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial t} - \frac{3}{2} h \frac{\partial h}{\partial x} + \frac{1}{4}\frac{\partial^3 h}{\partial x^3}\right) - \frac{3}{4}\frac{\partial^2 h}{\partial y^2} = 0$$

$$\frac{3}{4}\frac{\partial^2 h}{\partial x^2} = \frac{3}{4}\frac{\partial^2 h}{\partial y^2}$$