401-3382-25L: Log-Sobolev Inequalities and Markov Semigroups

Lecture 5 – Brascamp-Lieb inequality and KLS

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Key concepts:

- Brascamp-Lieb inequality
- Isoperimetric inequality
- Relating isoperimetric inequality and Poincaré inequality
- Isoperimetric inequality via localization lemma
- Isoperimetric inequality via stochastic localization

The main conceptual takeaway from the Bakry-Émery criterion and Gamma calculus is that in Euclidean space, having $\nabla^2 V \succeq \rho \mathbb{I}_n$ is sufficient for $\mu \propto e^{-V}$ to have dimension independent Poincaré inequality. We look for forms of Poincaré inequality which requires weaker assumptions than $\nabla^2 V \succeq \rho \mathbb{I}_n$.

5.1 Brascamp-Lieb inequality

Theorem 5.1.1 (Brascamp-Lieb inequality [BL02], also known as Hessian Poincaré)**.** *For a probability measure $d\mu = e^{-V}dx$ on $\mathbb{R}^n$ for which the smooth potential $V : \mathbb{R}^n \to \mathbb{R}$ is strictly convex, then*

$$\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}^n} \left\langle (\nabla^2 V)^{-1} \nabla f, \nabla f \right\rangle d\mu,$$

for every smooth compactly supported function f on $\mathbb{R}^n$.

Remark 1. *In this Euclidean context, for the Langevin semigroup, we know from the last lecture that the curvature dimension condition $CD(\rho, \infty)$ is equivalent to $\nabla^2 V \succeq \rho \mathbb{I}_n$. Hence, if $\rho > 0$, we have*

$$\left\langle \left(\nabla^2 V\right)^{-1} \nabla f, \nabla f \right\rangle \leq \frac{1}{\rho} \left|\nabla f\right|^2.$$

So the Brascamp-Lieb inequality improves the Poincaré inequality in Theorem 4.2.1.

The original proof in [BL02] is via induction on n starting from the special case $n = 1$. Here we present a proof via semigroups, the same idea was presented in Theorem 4.9.1 of [BGL+14].

Proof of Theorem 5.1.1. The proof is only sketched. Let $L = \Delta - \nabla V \cdot \nabla$ be the Langevin semigroup. W.l.o.g, we may assume $\mathbb{E}_\mu f = 0$. First, we assume that a (sufficiently regular) solution to the following Poisson-type equation exists

$$Lg = f.$$

Then we have

$$\begin{aligned}
\mathbb{E}_\mu(f^2) &= \int f^2 d\mu \\
&\overset{(i)}{=} \int f Lg d\mu \\
&\overset{(ii)}{=} \int \langle \nabla f, \nabla g \rangle \\
&= \int \left\langle \left(\nabla^2 V\right)^{-\frac{1}{2}} \nabla f, \left(\nabla^2 V\right)^{\frac{1}{2}} \nabla g \right\rangle,
\end{aligned}$$

(i) follows from $Lg = f$, (ii) follows from integration by parts. Applying Cauchy-Schwarz inequality, we have

$$\int f^2 d\mu \leq \left(\int \left\langle \left(\nabla^2 V\right) \nabla g, \nabla g \right\rangle d\mu \right)^{\frac{1}{2}} \left(\int \left\langle \left(\nabla^2 V\right)^{-1} \nabla f, \nabla f \right\rangle \right)^{\frac{1}{2}}. \tag{5.1}$$

From Lemma 3 in the last lecture (Gamma-calculus for Langevin semigroup), we know that

$$\Gamma_2(g) = \mathrm{trace}\left(\left(\nabla^2 g\right)^2 \right) + \left\langle \nabla^2 V \nabla g, \nabla g \right\rangle \geq \left\langle \nabla^2 V \nabla g, \nabla g \right\rangle.$$

It remains to relate $\Gamma_2(g)$ back to $\int f^2 d\mu$. Because $\int Lg^2 = 0$, we have (recall $\Gamma(g) = \frac{1}{2}\left(Lg^2 - 2gLg\right)$)

$$\int \Gamma(g) d\mu = -\int g Lg d\mu.$$

Then (recall $\Gamma_2(g) = \frac{1}{2}\left(L\Gamma(g) - 2\Gamma(g, Lg)\right)$), we have

$$\begin{aligned}
\int \Gamma_2(g) d\mu &= -\int \Gamma(g, Lg) d\mu \\
&= \frac{1}{2} \int g L^2 g + (Lg)^2 d\mu \\
&= \int (Lg)^2 d\mu \\
&= \int f^2 d\mu.
\end{aligned}$$

Plugging the bound $\langle \nabla^2 V \nabla g, \nabla g\rangle \leq \Gamma_2(g) = \int f^2 d\mu$ into Eq. (5.1), we obtain

$$\int f^2 d\mu \leq \int \left\langle \left(\nabla^2 V\right)^{-1} \nabla f, \nabla f \right\rangle,$$

which is what we wanted. Finally, we show that a solution to $Lg = f$ can be constructed via semigroups. Define

$$u_t(x) := -\int_0^t P_s f(x) ds.$$

By definition, u_t satisfies

$$\partial_t u_t = -P_t f.$$

Then

$$\begin{aligned} Lu_t &= L\int_0^t -P_s f ds \\ &= \int_0^t -LP_s f ds \\ &= -[P_s f]_0^t \\ &= f - P_t f. \end{aligned}$$

Now P_t is ergodic with respect to μ and $\mathbb{E}_\mu f = 0$, letting $t \to \infty$, we obtain $P_t f = 0$. Hence, we conclude that u_∞ (after checking reguarlity) constitutes a solution to the equation $Lu = f$. □

5.2 Isoperimetric inequality

Give a probability measure μ on $\mathbb{R}^n$. Fix the distance $\mathbf{d}(x, y) = \|x - y\|_2$ to be the Euclidean distance. In general, the isoperimetric problem for μ is asking the question: among all sets of a given measure, which sets have the minimal perimeter?

To define the perimeter precisely, we adopt the exterior Minkowski content.

exterior Minkowski content as a boundary measure For every measurable subset A, we let

$$\mu_+(A) = \lim\inf_{r\to 0^+} \frac{\mu(A_r) - \mu(A)}{r},$$

where $A_r = \{x \in \mathbb{R} \mid \mathbf{d}(x, A) \leq r\}$ is the r-neighborhood of $A \subset \mathbb{R}^n$.

Isoperimetric inequality We say a measure μ satisfies an *isoperimetric inequality* with constant C if

$$\min\{\mu(A), 1-\mu(A)\} \leq C\mu_+(A) \tag{5.2}$$

for every measurable set A. The smallest C such that the above holds is called the isoperimetric constant of μ, denoted ψ_μ.

Relating Poincaré inequality with isoperimetric inequality Recall that we say μ satisfies a Poincaré inequality with constant $C_P(\mu)$ if

$$\mathrm{Var}_\mu(f) \leq C_P(\mu)\mathbb{E}_\mu |\nabla f|^2,$$

for all f such that both sides are well defined. Then we have the following theorem relating Poincaré inequality with isoperimetric inequality.

Theorem 5.2.1. *We have*

$$C_P(\mu) \leq 4\psi_\mu^2.$$

Remark that the converse of Theorem 5.2.1 is not true in general, even if we allow for additional universal constant factors. However, under the assumption that μ is log-concave, for example, Buser's inequality provides the converse up to universal constants [Bus82].

To prove the above result, we need the co-area formula (see e.g. Lemma 3.2 in [BH97]), which is essentially a way to rewrite an integral.

Lemma 1 (Co-area formula)**.** *For any Lipschitz function on $\mathbb{R}^n$, we have*

$$\int |\nabla f|\, d\mu \geq \int_{-\infty}^{\infty} \mu_+\left(\{x \in \mathbb{R}^n \mid f(x) > t\}\right) dt.$$

In other words, we rewrite the integral as an integral over a single parameter t which specifies the level sets of f. See Lemma 3.2 in [BH97] for a proof. Remark equality also holds under some additional properties of μ, such as nonsingularity.

Using the co-area formula, we can write the isoperimetric inequality (5.2) as a $\mathbb{L}^1$-variant of Poincaré inequality.

Lemma 2. *The following two statements are equivalent*

(a) μ *satisfies the isoperimetric inequality in Eq.* (5.2)

(b) For all f,

$$\min_{c\in\mathbb{R}} \int |f-c|\, d\mu \leq C \int |\nabla f|\, d\mu,$$

where $|\nabla f|$ is defined as

$$|\nabla f(x)| := \lim \sup_{\mathbf{d}(x,y)\to 0^+} \frac{|f(x)-f(y)|}{\mathbf{d}(x,y)}.$$

Proof of Lemma 2. $(a) \implies (b)$: w.l.o.g, we may assume median$(f) = 0$, that is $\mathbb{P}_\mu(f \geq 0) \geq \frac{1}{2}$ and $\mathbb{P}_\mu(f \leq 0) \geq \frac{1}{2}$. Applying co-area formula in Lemma 1, we have

$$\begin{aligned}
\int |\nabla f|\, d\mu &\geq \int_{-\infty}^{+\infty} \mu_+ \left(\{f > t\}\right) dt \\
&\overset{(i)}{\geq} \frac{1}{C} \int_{-\infty}^{+\infty} \min\left\{\mu(\{f > t\}), 1-\mu(\{f>t\})\right\} dt \\
&\overset{(ii)}{=} \frac{1}{C} \int_0^{+\infty} \mu(\{f > t\})dt + \int_{-\infty}^{0} \mu(\{f \leq t\})dt \\
&= \frac{1}{C}\mathbb{E}\,|f|\,.
\end{aligned}$$

(i) applies isoperimetric inequality. (ii) follows because median$(f) = 0$. Finally, $\min_{c\in\mathbb{R}} \int |f-c|\, d\mu$ is achieved at the median of f.
$(b) \implies (a)$: take f_n to be a sequence of soft indicators of a set A as follows

$$f_n = \left(1 - \frac{1}{\epsilon_n}\mathbf{d}(x, A_{\epsilon_n})\right)_+,$$

Take $\epsilon_n = \frac{1}{n}$, then

$$\frac{\mu(A_{\epsilon_n} \setminus A)}{\epsilon_n} \to \mu_+(A),$$

and

$$f_n \to_{n\to\infty} \mathbf{1}_A.$$

Taking limsup of (b), we obtain (a). $\square$

Proof of Theorem 5.2.1. We start with an isoperimetric inequality, and we want to prove a Poincaré inequality. We can write

$$f = f_+ + f_-,$$

where $f_+ = f \cdot \mathbf{1}_{f\geq 0}$ and $f_- = f \cdot \mathbf{1}_{f<0}$. We deal with the positive and the negative parts separately.

Applying isoperimetric inequality to f_+, we obtain

$$\begin{aligned}\int f_+^2 d\mu &\leq \psi_\mu \int \left|\nabla(f_+^2)\right| d\mu \\ &= 2\psi_\mu \int f_+ \left|\nabla f_+\right| d\mu \\ &\leq 2\psi_\mu \left(\int f_+^2 d\mu\right)^{\frac{1}{2}} \left(\left|\nabla f_+\right|^2\right)^{\frac{1}{2}},\end{aligned}$$

where the last inequality follows from Cauchy-Schwarz. Rearranging the above, we obtain taht

$$\int f_+^2 d\mu \leq 4\psi_\mu^2 \int \left|\nabla f\right|^2 \mathbf{1}_{f>0} d\mu.$$

Similar we obtain that

$$\int f_-^2 d\mu \leq 4\psi_\mu^2 \int \left|\nabla f\right|^2 \mathbf{1}_{f<0} d\mu.$$

Summing the two equations above, we obtain the desired Poincaré inequality. □

5.3 Diameter isoperimetric inequality

Strictly positive curvature lower bound is not necessary for proving a dimension-free isoperimetric inequality. Here we show that log-concavity and diameter upper bound suffice.

log-concave We say a measure μ on $\mathbb{R}^n$ is log-concave, if

$$\mu(\lambda x + (1-\lambda)y) \geq \mu(x)^\lambda \mu(y)^{1-\lambda}, \forall x, y \in \mathbb{R}^n, \lambda \in [0,1].$$

For a compact set $K \subseteq \mathbb{R}^n$, define its diameter $\mathrm{diam}(K) = \max\left\{\mathbf{d}(x,y) \mid x, y \in K\right\}$.

Theorem 5.3.1 (Diameter isoperimetric inequality)**.** *Suppose μ is a log-concave probability measure supported on a compact convex set K, such that $diam(K) \leq D$ with $D > 0$. Then for any partition of $\mathbb{R}^n$ into measurable sets S_1, S_2, S_3, we have*

$$\pi(S_3) \geq \frac{2\mathbf{d}(S_1, S_2)}{D} \min\left\{\pi(S_1), \pi(S_2)\right\}.$$

The convex set version of it was proved in [DF91], see also [KLS95]. Its proof relies on the following convex localization lemma [LS93]. We adopt the simplified notation in [Vem05].

Lemma 3 (Convex localization lemma). *Let $g, h : \mathbb{R}^n \to \mathbb{R}$ be lower semi-continuous integrable functions such that*

$$\int_{\mathbb{R}^n} g(x)dx > 0, \text{ and } \int_{\mathbb{R}^n} h(x)dx > 0.$$

Then there exists two points $a, b \in \mathbb{R}^n$ and a linear function $\ell : [0,1] \to \mathbb{R}_+$ such that

$$\int_0^1 \ell(t)^{n-1} g((1-t)a + tb)dt > 0 \text{ and } \int_0^1 \ell(t)^{n-1} h((1-t)a + tb)dt > 0.$$

Essentially, the lemma allows us to reduce the proof of two joint integral inequality in n-dimensional space to the proof of two joint integral inequality in 1-dimension. The main proof idea is to reduce the dimension of the space where we integrate iteratively, while keeping the two integrals to hold jointly, until the space becomes a needle. See [Vem05].

Assuming Lemma 3, we can reduce the proof of isoperimetric inequality in n-dimensional space to the proof of an isoperimetric inequality in 1-dimension for a log-concave measure supported on a set with diameter D. Note that the operations in Lemma 3 never expand the diameter D. See [Vem05] for a complete proof.

5.4 KLS conjecture and stochastic localization

If we assumes that the covariance of a logconcave measure μ is bounded in the place of diameter bound, then we may prove a slightly better isoperimetric inequality. This is capture by the following Kannn-Lovász-Simonovits conjecture [KLS95] (see also the survey [LV18]).

Conjecture 1 (Kannn-Lovász-Simonovits [KLS95]). *There exists a universal constant $c > 0$, such that for any log-concave probability measure μ on $\mathbb{R}^n$ which is isotropic ($\mathbb{E}_{X\sim\mu}[X] = 0, \mathrm{Cov}_{X\sim\mu}(X) = \mathbb{I}_n$), its isoperimetric constant satisfies*

$$\psi_\mu \le c.$$

The general problem is still open. The current best bound is $c\sqrt{\log n}$ due to Klartag [Kla23].

Theorem 5.4.1 (Klartag [Kla23]). *There exists a universal constant $c > 0$, such that for any log-concave probability measure μ on $\mathbb{R}^n$ which is isotropic ($\mathbb{E}_{X\sim\mu}[X] = 0, \mathrm{Cov}_{X\sim\mu}(X) = \mathbb{I}_n$), its isoperimetric constant satisfies*

$$\psi_\mu \le c\sqrt{n}.$$

Remark 2. 1. *According to the diameter isoperimetric inequality in Theorem 5.3.1, we have $\psi_\mu = cD$ if μ is supported on a convex set of diameter D.*

- *The convex localization based proof can be extended to the case $\mathrm{Cov}_{X\sim\mu}(X) = \mathbb{I}_n$, then the covariance of the 1-dimensional measure $\leq \sqrt{n}$. We obtained a bound $\psi_\mu \leq c\sqrt{n}$. Intuitively, for logconcave measure with covariance $\mathbb{I}_n$, Chebyshev's inequality also implies that the mass should concentrate inside a ball of radius $\sqrt{n}$.*
- *The convex localization based proof can also be extended to the case where μ is α-strongly logconcave, that is, $-\nabla^2 \log(\mu) \succeq \alpha \mathbb{I}_n$.*

2. *For a general measure with $\mathbb{E}_{X\sim\mu}[X] = 0, \mathrm{Cov}_{X\sim\mu}(X) = A$, after rescaling, the conjecture states that*

$$\psi_\mu \leq c \left\|A\right\|_2 .$$

The set of logconcave measures contains all uniform measure on all kinds of convex sets. Without knowing the exact form of these convex sets, we don't have a lot of tools to prove an isoperimetric inequality in high dimension. At a high level, the main proof idea is reduce to the problem to a much simpler problem that we are familiar with.

- In diameter isoperimetric inequality and convex localization lemma, the idea is to reduce the high dimension problem to many 1-dimensional problems.
- Here, we use Eldan's stochastic localization to reduce the problem to many Gaussian-like strongly logconcave isoperimetric problems.

5.4.1 Eldan's stochastic localization

TODO

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