Exercise 1.1.
Compute the Fourier transform of the following functions, paying attention to the spaces where they belong:
(a) $f(x) = e^{-|x|}$, $f \in L^1(\mathbb{R})$.

(b) $f(x) = \frac{\sin x}{x}$, $f \in L^2(\mathbb{R})$.

(c) $f(x) = e^{-\frac{1}{2}|x|^2}$, $f \in \mathcal{S}(\mathbb{R}^n)$.

Exercise 1.2. ★
In this exercise we will show the existence of a smooth partition of unity subordinate to a finite cover of a compact set.
(a) Prove that the function $g : \mathbb{R}^n \to \mathbb{R}$,

$$g(x) = \begin{cases} \exp\left(-\frac{1}{(1-|x|^2)^2}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

is smooth, nonnegative, and has support in $\overline{B_1(0)}$.

(b) Let $K \subset \mathbb{R}^n$ be a compact set, and $K \subset U \subset \mathbb{R}^n$ be an open set. Show that there exists a function $\Theta \in C_c^\infty(U)$ such that $0 \leq \Theta \leq 1$ everywhere and $\Theta \equiv 1$ on K.

Hint: use convolution with a mollifier $g_\epsilon(x) := \epsilon^{-n} g(\epsilon^{-1}x)$ for $\epsilon > 0$ sufficiently small.

(c) Let $K \subset \mathbb{R}^n$ be compact and $U_1, \ldots, U_p$ open subsets of $\mathbb{R}^n$ which cover K. Show that there exist functions $\Theta_1, \ldots, \Theta_p$ such that for every $i \in \{1, \ldots, p\}$, $\Theta_i \in C_c^\infty(U_i)$, $0 \leq \Theta_i \leq 1$ everywhere, and such that $\Theta_1 + \cdots + \Theta_p \equiv 1$ on K.

Exercise 1.3. ★
Prove that for every $n \geq 1$ the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is separable.
Hint: approximate a function $f \in \mathcal{S}(\mathbb{R}^n)$ in each small cube by a rational number, and regularize the resulting function by mollification.

Exercise 1.4.
Show the following *baby* version of the Poincaré inequality in the Schwartz space: for every $f \in \mathcal{S}(\mathbb{R}^n)$, with $n \geq 1$, and for every $R > 0$, it holds that

$$\|f\|_{L^1(B_R(0))} \leq 2R\|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

Exercise 1.5. ♣
Show that there does not exist a function $\delta \in L^1(\mathbb{R}^n)$ (with respect to the Lebesgue measure) such that $\delta \star f = f$ for every $f \in \mathcal{S}(\mathbb{R}^n)$.