Exercise 2.1. ★
Show that the following spaces admit natural structures of Fréchet spaces:
(a) $L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

(b) $C^\infty(\overline{\Omega})$, where Ω is a bounded open set in $\mathbb{R}^n$. Recall that this space is defined as the space of functions $f \in C^\infty(\Omega)$ such that each $\partial^\alpha f$ extends continuously to $\overline{\Omega}$.

(c) $C^\infty(\mathbb{R}^n)$, where we do not assume any global bound on f or its derivatives.

(d) The Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Exercise 2.2. ★
Show that $C^\infty_c(\mathbb{R}^n)$ does **not** admit the structure of a Fréchet space. More precisely, prove the following statement: suppose that $(\mathcal{N}_p)_{p\in\mathbb{N}}$ is a family of seminorms on $C^\infty_c(\mathbb{R}^n)$ and d denotes the induced distance. Assume that for every sequence $(\varphi_j)_{j\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^n)$ converging to some $\varphi \in C^\infty_c(\mathbb{R}^n)$ with respect to d, also $\varphi_j \to \varphi$ *pointwise*[1]. Then show that $(C^\infty_c(\mathbb{R}^n), d)$ cannot be complete.

Exercise 2.3.
(a) Show that the function $f(x) = e^x$ is not a tempered distribution, i.e. that there exists no $T \in \mathcal{S}'(\mathbb{R})$ such that $\langle T, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x$ for every $\varphi \in C^\infty_c(\mathbb{R})$.

(b) Show that the function $g(x) = e^x\cos(e^x)$ does define a tempered distribution (in the above sense).

Exercise 2.4.
Let $T \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\varphi \equiv 0$ on $\operatorname{supp} T$. Is it true that $\langle T, \varphi\rangle = 0$?

Exercise 2.5.
For $\lambda > 0$, let $T_\lambda : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\langle T_\lambda, \varphi\rangle := \lim_{\varepsilon\to 0} \int_{(-\infty,-\varepsilon]\cup[\lambda\varepsilon,\infty)} \frac{\varphi(t)}{t}\,\mathrm{d}t.$$

Show that T_λ is a tempered distribution and relate it to $T_1 = \mathrm{p.v.}\,\frac{1}{t}$.

[1] One has to assume some kind of compatibility between the norms $\mathcal{N}_p$ and the usual convergence of functions, otherwise $C^\infty_c(\mathbb{R}^n)$ is just an abstract vector space and one can put even a Banach structure on it. In fact, it is enough to just require that $\varphi_j \rightharpoonup \varphi$ *in the sense of distributions*.

Exercise 2.6. ♣
Let $1 < p < \infty$ and $(u_j)_{j\in\mathbb{N}} \subset L^p(\mathbb{R})$ be a sequence of functions such that $u_j \rightharpoonup u$ in $\mathcal{S}'(\mathbb{R})$ for a function $u \in L^p(\mathbb{R})$.
(a) Show that

$$u_j \rightharpoonup u \text{ in } L^p(\mathbb{R}^n) \qquad \Longleftrightarrow \qquad \|u_j\|_{L^p} \leq C \text{ for some constant } C > 0.$$

(b) For each $1 < p < \infty$, construct one such sequence that does not satisfy the two equivalent conditions from part (a), that is, $u_j, u \in L^p(\mathbb{R})$ such that $u_j \rightharpoonup u$ in $\mathcal{S}'$ but not in L^p.

Exercise 2.7.
Consider the evaluation map $\mathrm{ev}_0 : C_b^0(\mathbb{R}) \to \mathbb{R}$, $\mathrm{ev}_0(\varphi) = \varphi(0)$, where $C_b^0(\mathbb{R})$ denotes the Banach space of all continuous and bounded functions on $\mathbb{R}$.
(a) Show that there exists a linear bounded extension $T : L^\infty(\mathbb{R}) \to \mathbb{R}$ of ev_0.

(b) Show that any such $T \in L^\infty(\mathbb{R})^*$ does not correspond to any function in $L^1(\mathbb{R})$. As a result, the embedding $L^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R})^{**} = L^\infty(\mathbb{R})^*$ is far from being surjective.