Exercise 3.1. ★
Let $T \in \mathcal{E}'(\mathbb{R}^n)$ and let $p \in \mathbb{N}$ be the order of T. Consider moreover $\varphi \in C^\infty(\mathbb{R}^n)$, such that $\partial^\alpha \varphi = 0$ on $\operatorname{supp} T$ for any α such that $|\alpha| \leq p$. Show that

$$\langle T, \varphi \rangle = 0.$$

Hint: denoting $K := \operatorname{supp} T$ and $K_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq \delta\}$, define for every small enough $\varepsilon > 0$ the function $\psi_\varepsilon := \mathbb{1}_{K_{2\varepsilon}} \star \chi_\varepsilon$, where $(\chi_\varepsilon)_{\varepsilon>0}$ is a standard family of mollifiers, and given any $\varphi \in C^\infty(\mathbb{R}^n)$, investigate the behavior of $\langle T, \varphi\psi_\varepsilon \rangle$ as $\varepsilon \to 0$.

Exercise 3.2. ★
We have seen in the lecture that for every $T \in \mathcal{S}'(\mathbb{R}^n)$, there is a well-defined map

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \longmapsto T \star \varphi \in C^\infty(\mathbb{R}^n).$$

(a) Show that this map is continuous as a linear map between Fréchet spaces.

(b) Show that, if in addition $T \in \mathcal{E}'(\mathbb{R}^n)$, then $T \star \varphi \in \mathcal{S}(\mathbb{R}^n)$.

(c) In this case, prove that moreover the map is continuous into the space $\mathcal{S}(\mathbb{R}^n)$.

Exercise 3.3.
Recall the definition of the translation operators: for $a \in \mathbb{R}^n$ we first define $\tau_a\varphi(x) = \varphi(x-a)$ for functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and then we define by duality $\langle \tau_a T, \varphi \rangle := \langle T, \tau_{-a}\varphi \rangle$ for $T \in \mathcal{S}'(\mathbb{R}^n)$.
(a) ★ Prove that $\forall T \in \mathcal{S}'(\mathbb{R}^n), \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\forall a \in \mathbb{R}^n$ it holds:

$$\tau_a(T \star \varphi) = (\tau_a T) \star \varphi = T \star \tau_a\varphi.$$

(b) Let $U : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be a linear continuous map commuting with translations, that is, such that for any $a \in \mathbb{R}^n$, $U \circ \tau_a = \tau_a \circ U$. Prove that there exists a $T \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$U\varphi = T \star \varphi \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Exercise 3.4.
(a) Determine all the tempered distributions $T \in \mathcal{S}'(\mathbb{R})$ such that $tT = 1$ (here t is the independent variable of $\mathbb{R}$ and 1 denotes the constant function 1, seen as a distribution).

(b) Does there exist any tempered distribution $S \in \mathcal{S}'(\mathbb{R})$ such that $t^2 S = 1$?

Exercise 3.5.
(a) Given a rotation $A \in \mathsf{SO}(n)$, define by duality the rotation operator $\mathsf{R}_A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ for tempered distributions, extending the rotation operator $\mathsf{R}_A f(x) := f(Ax)$ of functions. How is the Fourier transform of $\mathsf{R}_A T$ related to $\widehat{T}$?

(b) Given a scalar $\lambda > 0$, define by duality the dilation operator $\mathsf{D}_\lambda : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ for tempered distributions, extending the dilation operator $\mathsf{D}_\lambda f(x) := f(\lambda x)$ of functions. How is the Fourier transform of $\mathsf{D}_\lambda T$ related to $\widehat{T}$?

(c) Show that if $T \in \mathcal{S}'(\mathbb{R}^n)$ is radially symmetric then so is $\widehat{T}$. Show that if T is α-homogeneous then $\widehat{T}$ is β-homogeneous for some $\beta \in \mathbb{R}$. What is β?

(d) Show that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ and is radially symmetric (that is, $\mathsf{R}_A f = f$ for every $A \in \mathsf{SO}(n)$) and α-homogeneous (that is, $\mathsf{D}_\lambda f = \lambda^\alpha f \ \forall \lambda > 0$), then $f(x) = c|x|^\alpha$ almost everywhere, for some $c \in \mathbb{R}$.