Exercise 4.1. ★
Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$, and recall that we defined $T \star S, S \star T \in \mathcal{S}'(\mathbb{R}^n)$ as

$$\langle T \star S, \varphi \rangle = \langle T, \check{S} \star \varphi \rangle_{\mathcal{S}',\mathcal{S}} \qquad \text{and} \qquad \langle S \star T, \varphi \rangle = \langle S, \check{T} \star \varphi \rangle_{\mathcal{E}',C^\infty}$$

respectively, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.
(a) Let $(\chi_\varepsilon)_{\varepsilon>0}$ be a sequence of mollifiers as usual, and for $S \in \mathcal{E}'(\mathbb{R}^n)$, define $S_\varepsilon := S \star \chi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$. Show that, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$S_\varepsilon \star \varphi \xrightarrow{\varepsilon \to 0} S \star \varphi \qquad \text{in } \mathcal{S}(\mathbb{R}^n).$$

(b) Show that $S \star T = T \star S$ whenever $T \in \mathcal{S}'(\mathbb{R}^n)$ and $S \in \mathcal{E}'(\mathbb{R}^n)$. To do this, first prove it for S_ε in place of S, and then use the first part (applied to $\check{S}$) to conclude.

Exercise 4.2.
For each $0 < \alpha < n$, show that the function $f(x) = |x|^{-\alpha}$ defines a tempered distribution and compute its Fourier transform.
Hint: first consider $\alpha > n/2$, show that $\hat{f}$ is an L^1_{loc} function and apply Exercise 3.5 to deduce that $\hat{f}(\xi) = \gamma|\xi|^\beta$ for some $\beta, \gamma \in \mathbb{R}$ with β explicit. In order to find γ, test against a Gaussian $e^{-|x|^2/2}$, integrate in polar coordinates and relate the resulting expression to the Γ function. Argue for $\alpha < n/2$ using the inverse Fourier transform and finally for $\alpha = n/2$ by approximation.

Exercise 4.3.
Show that, for each $n \geq 2$ and $1 \leq i \leq n$, $\text{p. v.} \dfrac{x_i}{|x|^{n+1}}$ defines a tempered distribution and compute its Fourier transform.
Hint: use the previous exercise.

Exercise 4.4. ★
Recall that the distribution $S \in \mathcal{S}'(\mathbb{R}^4)$ defined by

$$\langle S, \varphi \rangle := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\varphi(x, |x|)}{|x|} \, \mathrm{d}x, \qquad \varphi \in \mathcal{S}(\mathbb{R}^4),$$

is a fundamental solution of the wave operator $\Box$. Show that, given $f \in \mathcal{E}'(\mathbb{R}^4)$, $u := S \star f$ is the only solution to $\Box u = f$ which is supported in $\mathbb{R}^3 \times (t_0, \infty)$ for some $t_0 > 0$.

Exercise 4.5. ★

(a) Show that the formal solution to the heat equation with initial data $f \in \mathcal{S}'(\mathbb{R}^n)$ obtained in the lecture,

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} \left(e^{-|\cdot|^2/4t} \star f \right)(x) \tag{†}$$

satisfies the initial condition in the following sense:

$$u(t,\cdot) \xrightarrow{t\to 0} f \qquad \text{in } \mathcal{S}'(\mathbb{R}^n). \tag{IC}$$

(b) Show rigorously that, if $u \in C^1(\mathbb{R}^+, L^1(\mathbb{R}^n))$ satisfies (IC) for some $f \in L^1(\mathbb{R}^n)$ and also

$$\partial_t \langle u, \varphi \rangle = \langle \Delta u, \varphi \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n),$$

then u must be given by the formula (†) and in particular it is unique in this class.