In this sheet, Exercises 5.1, 5.2 and 5.3 will deal with the following Cauchy problem for the wave equation:

$$\begin{cases} \Box u = u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u(\cdot,0) = f & \text{on } \mathbb{R}^n, \\ u_t(\cdot,0) = g & \text{on } \mathbb{R}^n. \end{cases} \tag{P}$$

Exercise 5.1. ★

The goal of this exercise is to solve the problem (P) by using the Fourier transform.

(a) Denoting by $\hat{u}(\xi,t)$ the Fourier transform of $u(\cdot,t)$ for each $t>0$, find the formal differential equation satisfied by $\hat{u}(\xi,t)$ and solve it taking into account the Fourier transforms of the initial conditions f and g.

(b) Show that, for $f,g \in \mathcal{E}'(\mathbb{R}^n)$, the formal expression obtained, $u(t)$, actually belongs to $\mathcal{S}'(\mathbb{R}^n)$ for every $t \geq 0$ and, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the function $t \in [0,\infty) \mapsto \langle u(t),\varphi\rangle \in \mathbb{R}$ is of class C^2 and satisfies

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle u(t),\varphi\rangle & = \langle \Delta u, \varphi\rangle \\ \langle u(0),\varphi\rangle & = \langle f,\varphi\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle u(t),\varphi\rangle & = \langle g,\varphi\rangle. \end{cases}$$

(c) In case $n=3$, show that $\forall t>0$

$$(2\pi)^{-3/2}\left\langle \mathcal{F}^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\right], \varphi\right\rangle = \frac{1}{4\pi t}\int_{\partial B_t} \varphi(x)\,\mathrm{d}\sigma(x) =: \langle R(t),\varphi\rangle$$

and deduce the following explicit expression for $u(t)$:

$$u(t) = R(t) \star g + \partial_t R(t) \star f.$$

Hint: compute the Fourier transform of the distribution $R(t)$.

Exercise 5.2. ★

Show that if u solves the Cauchy problem (P) for $f,g \in \mathcal{E}'(\mathbb{R}^n)$ and $u(t) \in \mathcal{E}'(\mathbb{R}^n)$ for each $t>0$, then u is unique.

Hint: fix a family of mollifiers (χ_ε) and show that $w_\varepsilon := w \star \chi_\varepsilon$ is a smooth solution of (P), where w is the difference of two solutions with the same initial data. Then consider the *energy*

$$E(w_\varepsilon,t) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla w_\varepsilon|^2 + |\partial_t w_\varepsilon|^2\,\mathrm{d}x$$

and show that $E(w_\varepsilon,t) = E(w_\varepsilon,0)$ for all $t>0$.

Exercise 5.3.
In this exercise we will relate the initial value problem (P) to the inhomogeneous wave equation, and then use Exercise 5.1 to find a fundamental solution of the D'Alembertian $\Box$ in arbitrary dimension supported in the future.
We look for a fundamental solution $S \in \mathcal{S}'(\mathbb{R}^n)$ that can be written as

$$\langle S, \psi \rangle = \int_0^\infty \langle v(t), \psi(\cdot, t) \rangle \, \mathrm{d}t$$

for some map $t \in [0, \infty) \mapsto v(t) \in \mathcal{S}'(\mathbb{R}^n)$.
(a) If we impose $\Box S = \delta$, show that v satisfies the Cauchy problem (P) and determine the initial conditions $f, g \in \mathcal{E}'(\mathbb{R}^n)$.

(b) Using Exercise 5.1, find an expression for S in arbitrary dimensions, and check that in dimension $n = 3$ it gives

$$\langle S, \psi \rangle = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\psi(x, |x|)}{|x|} \, \mathrm{d}x.$$

Exercise 5.4. ★
Consider the following Cauchy problem for the Schrödinger equation

$$\begin{cases} i\partial_t u = -\Delta u & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(0) = f & \text{in } \mathbb{R}^n, \end{cases} \tag{S}$$

where $f \in \mathcal{E}'(\mathbb{R}^n)$ is given. Solve the problem using the Fourier transform, and show that the formal solution obtained $t \in [0, \infty) \mapsto u(t) \in \mathcal{S}'(\mathbb{R}^n)$ is well-defined, continuous, satisfies $u(0) = f$, and solves (S) in the sense that

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n), \qquad t \mapsto \langle u(t), \varphi \rangle \text{ is in } C^1((0, \infty)) \quad \text{with} \quad i\frac{\mathrm{d}}{\mathrm{d}t} \langle u(t), \varphi \rangle = -\langle \Delta u(t), \varphi \rangle.$$

Moreover, show that for every $t > 0$,

$$u(t) = K_t \star f, \qquad \text{where } K_t(x) = \frac{1}{(4\pi i t)^{n/2}} e^{i|x|^2/4t}.$$

Hint: you may need to use the fact that, for $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$, the function $e^{-a|x|^2}$ is a Schwartz function and its Fourier transform is $(2a)^{-n/2} e^{-|\xi|^2/4a}$ (the same proof for $a \in \mathbb{R}^+$ works).

Exercise 5.5. ♣
Recall that, given an open set $\Omega \subset \mathbb{R}^n$, an integer $k \geq 0$ and $0 < \alpha \leq 1$, we let

$$C^{k,\alpha}(\overline{\Omega}) = \left\{ f \in C^k(\overline{\Omega}) : \|f\|_{C^{k,\alpha}(\overline{\Omega})} < +\infty \right\},$$

where

$$\|f\|_{C^{k,\alpha}(\overline{\Omega})} := \|f\|_{C^k(\overline{\Omega})} + \max_{|\beta| \leq k} \sup_{x \neq y \in \Omega} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha}.$$

(a) Show that $\| \cdot \|_{C^{k,\alpha}(\overline{\Omega})}$ defines a norm and that $C^{k,\alpha}(\overline{\Omega})$ is complete.

(b) Assume that Ω is bounded, and fix $0 < \gamma < \alpha \leq 1$. Show that the canonical embedding $C^{k,\alpha}(\overline{\Omega}) \hookrightarrow C^{k,\gamma}(\overline{\Omega})$ is compact.