Exercise 7.1.
Let V be a Banach space, $W \subset V$ a closed linear subspace and $v \in V \setminus W$. Show that there exists a linear functional $\ell \in V^*$ such that $\langle \ell, v\rangle \neq 0$ but $\langle \ell, w\rangle = 0$ for all $w \in W$. Moreover, show that one can choose ℓ such that $\langle \ell, v\rangle = \operatorname{dist}(v, W)$ and $\|\ell\| \leq 1$.

Exercise 7.2.
Prove that any closed linear subspace of a reflexive Banach space is also reflexive.

Exercise 7.3.
Let $(E, \|\cdot\|_E)$ be a Banach space. Prove that E is reflexive if and only if E^* is reflexive.

Exercise 7.4.
Let E be a Banach space, $J : E \to E^{**}$ the natural embedding into its bidual, and denote by B_E and $B_{E^{**}}$ the respective closed unit balls. The goal of this exercise is to prove the following lemma of Goldstine: that $J(B_E)$ is dense in $B_{E^{**}}$ with respect to the weak-* topology (of E^{**} with respect to E^*).
(a) Show that it is enough to prove that, given $\xi \in B_{E^{**}}$, $\varepsilon > 0$, an integer N, and linearly independent forms $\ell_1, \dots, \ell_N \in E^*$,

$$\exists\, z \in B_E \quad \text{s.t.} \quad |\langle \ell_i, z\rangle - \langle \xi, \ell_i\rangle| \leq \varepsilon \quad \text{for } i = 1, \dots, N. \tag{$\star$}$$

(b) Let $T : E \to \mathbb{R}^N$ be defined by $u \mapsto (\langle \ell_i, u\rangle)_{i=1}^N$. Show that T is surjective and that any map $\phi \in E^*$ vanishing on $\ker T$ is a linear combination of $\ell_1, \dots, \ell_N$.

(c) Show that given $\delta > 0$, one can find $y \in E$ with $\|y\| \leq 1 + \delta$ such that $\langle \ell_i, y\rangle = \langle \xi, \ell_i\rangle$ for $i = 1, \dots, N$.

Hint: start with any solution of the system and improve it with the help of Exercise 7.1.

(d) Conclude the proof of Goldstine's lemma.

Exercise 7.5.
We say that a Banach space $(E, \|\cdot\|)$ is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in E$,

$$\|x\| \leq 1, \quad \|y\| \leq 1 \quad \text{and} \quad \|x - y\| \geq \varepsilon \quad \Longrightarrow \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

In this exercise we will prove the Milman–Pettis theorem: every uniformly convex Banach space is reflexive. To do that, fix $\xi \in E^{**}$ with $\|\xi\| = 1$ and $\varepsilon > 0$, let $\delta > 0$ be the corresponding number from the uniform convexity condition, and choose $\ell \in E^*$ such that $\|\ell\| = 1$ and $\langle \xi, \ell \rangle > 1 - \delta/2$. Moreover let $V := \{\zeta \in E^{**} : \langle \zeta, \ell \rangle > 1 - \delta\}$.
(a) Show that the diameter of $V \cap J(B_E)$ is at most ε.

(b) Show that $B_\varepsilon(\xi) \cap J(B_E) \neq \varnothing$.

Hint: take any $J(x) \in J(B_E) \cap V$ and show that $\|J(x) - \xi\| \leq \varepsilon$ using Exercise 7.4.

(c) Conclude the proof of the Milman–Pettis theorem.

Exercise 7.6.
Let Ω be an open subset of $\mathbb{R}^n$.
(a) Show that $L^\infty(\Omega)$ is not separable.

(b) Prove that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for every $1 \leq p < +\infty$, and hence that $L^p(\Omega)$ is separable for $1 \leq p < \infty$.

Exercise 7.7.
Proof the so-called Littlewood inequality in a measure space (X, μ): given $1 \leq p_0 < p_1 \leq +\infty$ and $t \in (0, 1)$, define $p_t \in [1, +\infty]$ by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1};$$

then for any $f \in L^{p_0}(X, \mu) \cap L^{p_1}(X, \mu)$ it holds that

$$f \in L^{p_t}(X, \mu) \qquad \text{with} \qquad \|f\|_{L^{p_t}} \leq \|f\|_{L^{p_0}}^{1-t} \, \|f\|_{L^{p_1}}^{t}.$$