Exercise 8.1.
In this exercise we will show that the range of exponents of the Hausdorff–Young inequality is sharp. To do that, consider $f(x) = e^{-x^2/2}$ and fix some parameter $a > 0$ ($a = 10$ will do). Then for $N \in \mathbb{N}$ define

$$f_N(x) := \sum_{j=1}^{N} e^{ixja} f(x - ja).$$

(a) Compute $\hat{f}_N$.

(b) Show that for all $p \in [1, \infty]$ there is a constant $c > 0$ such that $\|f_N\|_{L^p(\mathbb{R})} \geq cN^{1/p}$ and $\|\hat{f}_N\|_{L^p(\mathbb{R})} \geq cN^{1/p}$.

(c) Show that for some constant $C > 0$, $\|f_N\|_{L^1(\mathbb{R})} \leq CN$ and $\|\hat{f}_N\|_{L^1(\mathbb{R})} \leq CN$ for all N.

(d) Show that for some constant $C' > 0$, $\|f_N\|_{L^\infty(\mathbb{R})} \leq C'$ and $\|\hat{f}_N\|_{L^\infty(\mathbb{R})} \leq C'$ for all N.

(e) Conclude that, if $p \in [1, \infty]$ is such that the Fourier transform is bounded from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$, then necessarily $p \leq 2$.

Exercise 8.2.
In this exercise we will prove the "integral Minkowski inequality": let (X, μ) and (Y, ν) be two σ-finite measure spaces[1] and let $f : X \times Y \to [0, \infty)$ be measurable with respect to the product measure. Show that for each $1 \leq p < +\infty$ it holds:

$$\left(\int_X \left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right)^p \mathrm{d}\mu(x)\right)^{1/p} \leq \int_Y \left(\int_X f(x,y)^p\,\mathrm{d}\mu(x)\right)^{1/p} \mathrm{d}\nu(y).$$

Hint: look at what inequality you get when $(Y, \nu) = (\{1, 2\}, \#)$ and try to replicate the proof of that inequality from Measure Theory.

[1]You can just take them to be measurable subsets of Euclidean space with the Lebesgue measure—we only need that the product measure is well defined and that Tonelli's theorem holds.

Exercise 8.3.
Fix $1 \leq p \leq \infty$ and suppose that $K : (0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfies the following two properties:

- K is homogeneous of degree -1, that is, for $\lambda > 0$, $K(\lambda x, \lambda y) = \lambda^{-1} K(x,y)$.
- it holds that $A_K := \int_0^\infty |K(1,y)| y^{-1/p} \, \mathrm{d}y < +\infty$.

We define the linear operator

$$(Tf)(x) := \int_0^\infty K(x,y) f(y) \, \mathrm{d}y;$$

show that $\|Tf\|_{L^p} \leq A_K \|f\|_{L^p}$.
Hint: write the function $(Tf)(x)$ as an integral of functions of x depending on some other parameter, and apply the integral Minkowski inequality.

Exercise 8.4.
(a) Show the following version of the Hardy inequality: given a measurable function $g : (0,\infty) \to \mathbb{R}$ and two real numbers $1 \leq p < \infty$ and $r > 0$,

$$\int_0^\infty \left(\int_0^x |g(y)| \, \mathrm{d}y \right)^p x^{-r-1} \, \mathrm{d}x \leq \left(\frac{p}{r} \right)^p \int_0^\infty (y|g(y)|)^p y^{-r-1} \, \mathrm{d}y.$$

Hint: deduce it from the estimate of Exercise 8.3.

(b) Obtain the following more common version of the Hardy inequality: if $u : [0,\infty) \to \mathbb{R}$ is an absolutely continuous function[2] with $u(0) = 0$, then for any $p > 1$ it holds

$$\int_0^\infty \left(\frac{|u(x)|}{x} \right)^p \mathrm{d}x \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty |u'(x)|^p \, \mathrm{d}x.$$

[2]This just means that u is the primitive of an L^1 function.

Exercise 8.5.

This exercise assumes familiarity with the Riesz representation theorem for measures and the Radon–Nikodym theorem.

The goal of this exercise is to give a different proof of the fact that the dual of L^p is $L^{p'}$ (where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$) using two classical theorems from Measure Theory instead of abstract Functional Analysis. For simplicity we deal with an open set $\Omega \subset \mathbb{R}^n$, which we write as an increasing union of bounded open sets $\Omega_1 \subset \Omega_2 \subset \cdots$. Let $\ell : L^p(\Omega) \to \mathbb{R}$ be a linear bounded functional.

(a) Show that for each j, ℓ naturally defines a linear bounded functional on $C_c(\Omega_j)$. Therefore, by the Riesz representation theorem, we get a *signed*[3] Radon measure ν_j on Ω_j.

(b) Show that for every j, $|\nu_j| \ll \mathcal{L}^n$. Hence, by the Radon–Nikodym theorem we get measurable functions $f_j \in L^1(\Omega_j)$ such that $\mathrm{d}\nu_j = f_j \mathrm{d}\mathcal{L}^n$. Show also that the functions f_j and f_{j+1} agree almost everywhere on Ω_j and hence they define a global function $f \in L^1_{\mathrm{loc}}(\Omega)$.

(c) Show that $f \in L^{p'}(\Omega)$ and conclude.

[3] A signed Radon measure ν is just the difference of two positive (usual) Radon measures ν^+ and ν^-; this decomposition is unique if ν^+ and ν^- are mutually orthogonal, and in this case we denote the total variation measure by $|\nu| := \nu^+ + \nu^-$.